

## Sample Problems

- Find the coordinates of all points where the circle  $x^2 + (y - 3)^2 = 50$  and the line  $y = -2x + 8$  intersect each other.
- Find the points where the circles  $(x + 4)^2 + (y + 1)^2 = 10$  and  $(x + 1)^2 + (y - 5)^2 = 25$  intersect each other.
  - Find the points where the circles  $(x + 2)^2 + (y + 2)^2 = 50$  and  $(x - 2)^2 + (y - 1)^2 = 25$  intersect each other.
- Consider the circles  $x^2 + y^2 = 36$  and  $(x - 5)^2 + y^2 = 16$ . Let  $C$  be the intersection of the two common tangent lines drawn to the circle.
  - Find the coordinates of  $C$ .
  - Consider one of the common tangent lines. Find the distance between the points of tangency.
- Find an equation of the circle that passes through the points  $A(0, 2)$ ,  $B(-2, -2)$ , and  $C(-8, -4)$ .

## Practice Problems

- Find the coordinates of all point(s) where the circle and the line intersect each other.
  - $(x + 5)^2 + (y - 7)^2 = 8$  and  $x + y = 6$
  - $(x - 2)^2 + (y + 1)^2 = 50$  and  $y = -x + 9$
  - $x^2 + (y + 4)^2 = 25$  and  $y = x - 24$
  - $(x - 3)^2 + (y + 1)^2 = 25$  and  $y = -x + 1$
- Find the coordinates of all points where the given circles intersect each other.
  - $(x - 1)^2 + (y - 2)^2 = 10$  and  $(x - 10)^2 + (y - 5)^2 = 40$
  - $(x - 3)^2 + y^2 = 20$  and  $x^2 + (y - 1)^2 = 50$
  - $x^2 + (y - 8)^2 = 26$  and  $(x - 7)^2 + (y - 3)^2 = 4$
- Consider the circles  $(x + 5)^2 + y^2 = 100$  and  $x^2 + y^2 = 49$ . Let  $C$  be the intersection of the two common tangent lines drawn to the circle.
  - Find the coordinates of  $C$ .
  - Consider one of the common tangent lines. Find the distance between the points of tangency.
- Let  $C_1$  and  $C_2$  be circles defined by  $x^2 + (y + 7)^2 = 1$  and  $x^2 + (y - 6)^2 = 16$ , respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.
- Given the three points, find an equation of the circle that contains all three points.
  - $P(0, 0)$ ,  $Q(0, 8)$ , and  $R(-6, 0)$ .
  - $A(-1, 6)$ ,  $B(-7, 12)$ , and  $C(-3, 0)$ .
  - $X(3, -7)$ ,  $Y(5, -1)$ , and  $Z(-3, -5)$ .

## Sample Problems - Answers

1.  $(-1, 10)$  and  $(5, -2)$       2. a)  $(-1, 0)$  and  $(-5, 2)$       b)  $(-1, 5)$  and  $(5, -3)$   
3. a)  $(15, 0)$       b)  $\sqrt{21}$       4.  $(x + 7)^2 + (y - 3)^2 = 50$

## Practice Problems - Answers

1. a)  $(-3, 9)$       b)  $(3, 6)$  and  $(9, 0)$       c) they don't intersect      d)  $(-1, 2)$  and  $(6, -5)$   
2. a)  $(4, 3)$       b)  $(7, 2)$  and  $(5, -4)$       c) the circles do not intersect  
3. a)  $\left(\frac{35}{3}, 0\right)$       b) 4      4.  $4\sqrt{10}$  or 12      5. a)  $(x + 3)^2 + (y - 4)^2 = 25$   
b)  $(x + 8)^2 + (y - 5)^2 = 50$       c)  $(x - 1)^2 + (y + 3)^2 = 20$

## Sample Problems - Solutions

1. Find the coordinates of all points where the circle  $x^2 + (y - 3)^2 = 50$  and the line  $y = -2x + 8$  intersect each other.

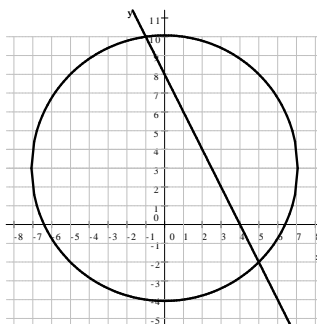
Solution: We need to solve the following system of equations:

$$\begin{cases} x^2 + (y - 3)^2 = 50 \\ y = -2x + 8 \end{cases}$$

We will use substitution. We substitute  $y = -2x + 8$  in the first equation and solve for  $x$ .

$$\begin{aligned} x^2 + (-2x + 8 - 3)^2 &= 50 & 5x^2 - 20x - 25 &= 0 \\ x^2 + (-2x + 5)^2 &= 50 & 5(x^2 - 4x - 5) &= 0 \\ x^2 + 4x^2 - 20x + 25 &= 50 & 5(x - 5)(x + 1) &= 0 \\ 5x^2 - 20x + 25 &= 50 & x_1 = 5 & \quad x_2 = -1 \end{aligned}$$

We now use the second equation to find the  $y$ -value belonging to the  $x$ -values we just obtained. If  $x = 5$ , then  $y = -2(5) + 8 = -2$ . If  $x = -1$ , then  $y = -2(-1) + 8 = 10$ . Thus the two points are  $(5, -2)$  and  $(-1, 10)$ . We check: both points should be on both the circle and the line.



2. a) Find the points where the circles  $(x + 4)^2 + (y + 1)^2 = 10$  and  $(x + 1)^2 + (y - 5)^2 = 25$  intersect each other.

Solution: We need to solve the following system:

$$\begin{cases} (x + 4)^2 + (y + 1)^2 = 10 \\ (x + 1)^2 + (y - 5)^2 = 25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations.

$$\begin{aligned} x^2 + 8x + 16 + y^2 + 2y + 1 &= 10 & x^2 + 2x + 1 + y^2 - 10y + 25 &= 25 \\ x^2 + 8x + y^2 + 2y + 17 &= 10 & x^2 + 2x + y^2 - 10y + 26 &= 25 \\ x^2 + 8x + y^2 + 2y &= -7 & x^2 + 2x + y^2 - 10y &= -1 \end{aligned}$$

We will multiply the second equation by  $-1$  and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned} x^2 + 8x + y^2 + 2y &= -7 \\ -x^2 - 2x - y^2 + 10y &= 1 \end{aligned}$$

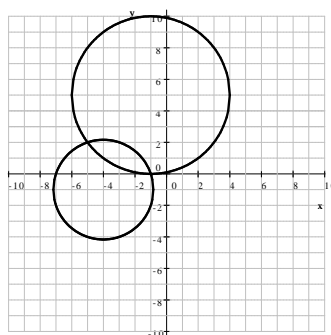
The sum of the two equations is

$$\begin{aligned} 6x + 12y &= -6 & \text{divide both sides by 6} \\ x + 2y &= -1 \\ x &= -2y - 1 \end{aligned}$$

We substitute this into the first equation and solve for  $y$ .

$$\begin{aligned}
 (x+4)^2 + (y+1)^2 &= 10 & \text{and} & & x &= -2y - 1 \\
 (x+4)^2 + (y+1)^2 &= 10 \\
 \left(\underbrace{-2y-1}_{x} + 4\right)^2 + (y+1)^2 &= 10 \\
 (-2y+3)^2 + (y+1)^2 &= 10 \\
 4y^2 - 12y + 9 + y^2 + 2y + 1 &= 10 \\
 5y^2 - 10y + 10 &= 10 \\
 5y^2 - 10y &= 0 \\
 5y(y-2) &= 0 & \implies & & y_1 &= 0 & & y_2 &= 2
 \end{aligned}$$

We now find the  $x$  values belonging to the  $y$ -values, using  $x = -2y - 1$ . If  $y_1 = 0$ , then  $x_1 = -2 \cdot 0 - 1 = -1$  and if  $y_2 = 2$ , then  $x_2 = -2 \cdot 2 - 1 = -5$ . Thus the two circles intersect at the points  $(-1, 0)$  and  $(-5, 2)$ .



b) Find the points where the circles  $(x+2)^2 + (y+2)^2 = 50$  and  $(x-2)^2 + (y-1)^2 = 25$  intersect each other.

Solution: We need to solve the following system:

$$\begin{cases} (x+2)^2 + (y+2)^2 = 50 \\ (x-2)^2 + (y-1)^2 = 25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations

$$\begin{aligned}
 x^2 + 4x + 4 + y^2 + 4y + 4 &= 50 & x^2 - 4x + 4 + y^2 - 2y + 1 &= 25 \\
 x^2 + 4x + y^2 + 4y &= 42 & x^2 - 4x + y^2 - 2y &= 20
 \end{aligned}$$

We will multiply the second equation by  $-1$  and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned}
 x^2 + 4x + y^2 + 4y &= 42 \\
 -x^2 + 4x - y^2 + 2y &= -20
 \end{aligned}$$

The sum of the two equations is

$$\begin{aligned}
 8x + 6y &= 22 & \text{divide both sides by 2} \\
 4x + 3y &= 11
 \end{aligned}$$

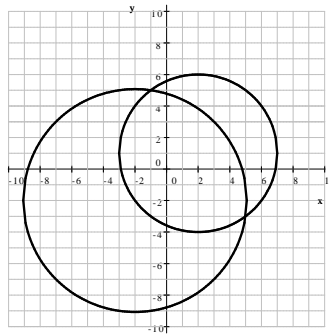
We now solve for  $y$

$$y = \frac{-4x + 11}{3} = -\frac{4}{3}x + \frac{11}{3}$$

We substitute this into the first equation and solve for  $x$ .

$$\begin{aligned}
 (x+2)^2 + (y+2)^2 &= 50 & \text{and} & & y &= -\frac{4}{3}x + \frac{11}{3} \\
 (x+2)^2 + \underbrace{\left(-\frac{4}{3}x + \frac{11}{3} + 2\right)}_y^2 &= 50 & & & \frac{11}{3} + 2 &= \frac{17}{3} \\
 (x+2)^2 + \left(\frac{-4x+17}{3}\right)^2 &= 50 \\
 (x+2)^2 + \frac{(-4x+17)^2}{3^2} &= 50 & \text{since } \left(\frac{a}{b}\right)^2 &= \frac{a^2}{b^2} \\
 (x+2)^2 + \frac{(-4x+17)^2}{9} &= 50 & \text{multiply by 9} \\
 9(x+2)^2 + (-4x+17)^2 &= 450 \\
 9(x^2 + 4x + 4) + 16x^2 - 136x + 289 &= 450 \\
 9x^2 + 36x + 36 + 16x^2 - 136x + 289 &= 450 \\
 25x^2 - 100x - 125 &= 0 \\
 25(x^2 - 4x - 5) &= 0 \\
 25(x-5)(x+1) &= 0 & \implies & & x_1 &= 5 & x_2 &= -1
 \end{aligned}$$

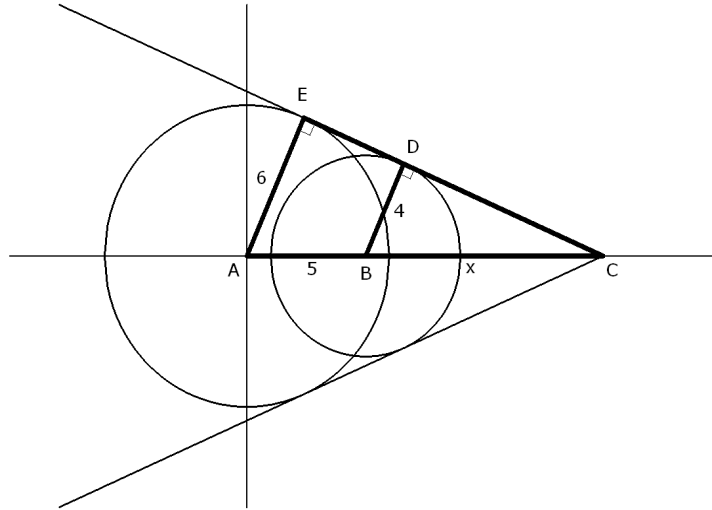
We now find the  $y$  values belonging to the  $x$ -values, using  $y = \frac{-4x+11}{3}$ . If  $x = 5$ , then  $y = \frac{-4(5)+11}{3} = -3$  and if  $x = -1$ , then  $y = \frac{-4(-1)+11}{3} = 5$ . Thus the two circles intersect at the points  $(-1, 5)$  and  $(5, -3)$ .



3. Consider the circles  $x^2 + y^2 = 36$  and  $(x - 5)^2 + y^2 = 16$ . Let  $C$  be the intersection of the two common tangent lines drawn to the circle.

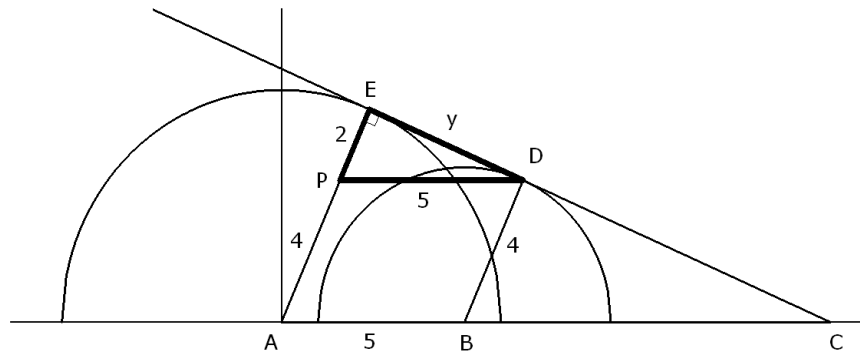
a) Find the coordinates of  $C$ .

Solution: Using the notation shown on the picture below, triangles  $ACE$  and  $BCD$  are similar. Based on that, we write the equation  $\frac{6}{4} = \frac{x+5}{x}$ . The solution of this equation is 10. Thus  $C$  is  $(15, 0)$ .



b) Consider one of the common tangent lines. Find the distance between the points of tangency.

Solution: We draw a horizontal line through the point  $D$ . Let us denote the intersection of  $AE$  and the horizontal line by  $P$ . The quadrilateral  $ABDP$  is a parallelogram since opposite sides are parallel. Consequently, opposite sides are equally long. This means that  $AP = 4$  and so  $PE = 2$ . Also,  $PD = 5$ . We can now compute  $y$  via the Pythagorean theorem:  $y = \sqrt{21}$ .



4. Find an equation of the circle that passes through the points  $A(0, 2)$ ,  $B(-2, -2)$ , and  $C(-8, -4)$ .

We will present two different solutions.

Solution 1. This solution is based on the following fact. The **perpendicular bisector of a line segment**  $AB$  is the set of all points in the plane that are equidistant to points  $A$  and  $B$ . Recall that the center of the circle is equidistant to all points on the circle. Recall that the line segment connecting two points on a circle is called a chord. The center of a circle is contained on the perpendicular bisector of any of its chords.

Step 1. Let us find the equation of the perpendicular bisector of line segment  $AB$ . It is a line that is perpendicular to  $AB$  and passes through its midpoint,  $M$ .

$$m_{AB} = \frac{y_A - y_B}{x_A - x_B} = \frac{2 - (-2)}{0 - (-2)} = \frac{4}{2} = 2 \quad \implies \quad m_{\text{perpendicular}} = -\frac{1}{2}$$

and the midpoint  $M$  can be found as

$$x_M = \frac{x_A + x_B}{2} = \frac{0 + (-2)}{2} = -1 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{2 + (-2)}{2} = 0 \quad M_{AB}(-1, 0)$$

Thus the perpendicular bisector of  $AB$  is a line with slope  $-\frac{1}{2}$ , passing through  $(-1, 0)$ . We can easily find this equation:

$$y = -\frac{1}{2}(x + 1) = -\frac{1}{2}x - \frac{1}{2}$$

Step 2. We repeat the entire procedure with line segment  $AC$ .

$$m_{AC} = \frac{y_A - y_C}{x_A - x_C} = \frac{2 - (-4)}{0 - (-8)} = \frac{6}{8} = \frac{3}{4} \quad \implies \quad m_{\text{perpendicular}} = -\frac{4}{3}$$

and the midpoint  $M$  can be found as

$$x_M = \frac{x_A + x_C}{2} = \frac{0 + (-8)}{2} = -4 \quad \text{and} \quad y_M = \frac{y_A + y_C}{2} = \frac{2 + (-4)}{2} = -1 \quad M_{AC}(-4, -1)$$

Thus the perpendicular bisector of  $AC$  is a line with slope  $-\frac{4}{3}$ , passing through  $(-4, -1)$ . We can easily find this equation:

$$\begin{aligned} y + 1 &= -\frac{4}{3}(x + 4) \\ y &= -\frac{4}{3}(x + 4) - 1 = -\frac{4}{3}x - \frac{16}{3} - 1 = -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

Step 3. The center of the circle lies on both perpendicular bisectors found in Steps 1 and 2. Thus it must be the intersection of those two lines. We find the intersection point by solving the system

$$\begin{aligned} y &= -\frac{1}{2}x - \frac{1}{2} \\ y &= -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

We can easily use substitution and obtain the equation

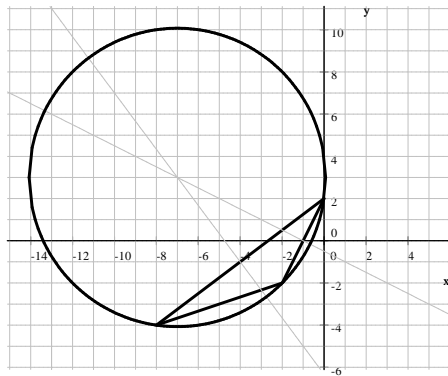
$$\begin{aligned} -\frac{1}{2}x - \frac{1}{2} &= -\frac{4}{3}x - \frac{19}{3} && \text{multiply by 6} \\ -3x - 3 &= -8x - 38 \\ 5x - 3 &= -38 \\ 5x &= -35 \\ x &= -7 && y = -\frac{1}{2}x - \frac{1}{2} = -\frac{1}{2}(-7) - \frac{1}{2} = \frac{7}{2} - \frac{1}{2} = 3 \end{aligned}$$

and obtain  $(-7, 3)$ .

Step 4. To find the radius, we need to compute the distance between  $(-7, 3)$  and any of the points  $A$ ,  $B$ , and  $C$ . Let us compute the distance of  $(-7, 3)$  from  $A$ .

$$r = \sqrt{(-7 - x_A)^2 + (3 - y_A)^2} = \sqrt{(-7 - 0)^2 + (3 - 2)^2} = \sqrt{50}$$

Thus the equation of the circle is  $(x + 7)^2 + (y - 3)^2 = 50$ .





Solution 2. Suppose that the circle has an equation

$$(x - k)^2 + (y - h)^2 = r^2$$

To find this equation, we must find the real numbers  $k$ ,  $h$ , and  $r^2$ . Since there are three unknown variables, we will need three equations. We obtain those by stating that points  $A$ ,  $B$ , and  $C$  are all on the circle.

$$\begin{array}{ll} 1) & (0 - k)^2 + (2 - h)^2 = r^2 & A(0, 2) \\ 2) & (-2 - k)^2 + (-2 - h)^2 = r^2 & B(-2, -2) \\ 3) & (-8 - k)^2 + (-4 - h)^2 = r^2 & C(-8, -4) \end{array}$$

We simplify the equations

$$\begin{array}{ll} 1) & k^2 + h^2 - 4h + 4 = r^2 & 1) & k^2 + h^2 - 4h + 4 = r^2 \\ 2) & k^2 + 4k + 4 + h^2 + 4h + 4 = r^2 & 2) & k^2 + 4k + h^2 + 4h + 8 = r^2 \\ 3) & k^2 + 16k + 64 + h^2 + 8h + 16 = r^2 & 3) & k^2 + 16k + h^2 + 8h + 80 = r^2 \end{array}$$

We can eliminate  $r$  and all expressions quadratic in  $k$  if we subtract the first equation from the second and the third equations. We will proceed carefully, adding the opposite instead of subtracting.

$$\begin{array}{ll} 1) & -k^2 - h^2 + 4h - 4 = -r^2 & 1) & -k^2 - h^2 + 4h - 4 = -r^2 \\ 2) & k^2 + 4k + h^2 + 4h + 8 = r^2 & 3) & k^2 + 16k + h^2 + 8h + 80 = r^2 \\ & \Downarrow & & \Downarrow \\ & 4k + 8h + 4 = 0 & & 16k + 12h + 76 = 0 \\ & k + 2h = -1 & & 4k + 3h = -19 \end{array}$$

We obtained a system in two variables. Also, this system is now completely linear.

$$\begin{array}{ll} k + 2h = -1 & \text{multiply by } -4 & \implies & -4k - 8h = 4 \\ 4k + 3h = -19 & & & 4k + 3h = -19 \end{array}$$

We add the two equations and obtain  $-5h = -15$  and so  $h = 3$ . From the equation  $k + 2h = -1$  we easily obtain the value  $k = -7$ . We now substitute  $k = -7$ ,  $h = 3$  into the first equation to obtain the value of  $r^2$ .

$$\begin{array}{ll} k^2 + (2 - h)^2 = r^2 & k = -7, h = 3 \\ (-7)^2 + (2 - 3)^2 = r^2 & \\ 50 = r^2 & \end{array}$$

And so the equation of the circle is  $(x + 7)^2 + (y - 3)^2 = 50$ . It is interesting how the two solutions presented here relate to each other.