Equations are a fundamental concept and tool in mathematics.

Definition: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $-x^2+3=4x-2$ is an equation. So is $x^2-y^2=2y-x+3$.

Definition: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true.

Example 1: Consider the equation $-x^2+3=4x-2$.

a) Verify that -2 is not a solution of the equation.

b) Verify that -5 is a solution of the equation.

Solution: a) If we substitute $x=-2$ into both sides of the equation $-x^2+3=4x-2$, then

$$\begin{aligned}
\text{the left-hand side is}\quad \text{LHS} &= -(-2)^2+3=-4+3=-1\\
\text{the right-hand side is}\quad \text{RHS} &= 4(-2)-2=-8-2=-10\\
-1 &\neq -10\\
\text{LHS} &\neq \text{RHS}
\end{aligned}$$

Since -2, when substituted into x in the equation, makes the statement of equality false, -2 is not a solution of this equation.

b) If we substitute $x=-5$ into both sides of the equation $-x^2+3=4x-2$, then

$$\begin{aligned}
\text{the left-hand side is}\quad \text{LHS} &= -(-5)^2+3=-25+3=-22\\
\text{the right-hand side is}\quad \text{RHS} &= 4(-5)-2=-20-2=-22\\
-22 &= -22\\
\text{LHS} &= \text{RHS}
\end{aligned}$$

Since -5, when substituted into x in the equation, makes the statement of equality true, -5 is a solution of this equation.

Example 2: Consider the equation $x^2-y^2=2y-x+3$.

a) Verify that the ordered pair $(1,-3)$ is not a solution of the equation.

b) Verify that the ordered pair $(-3,1)$ is a solution of the equation.

Solution: *Ordered pair* simply means that the order of the two numbers listed matters. $(1,-3)$ indicates that $x=1$ and $y=-3$. On the other hand, the ordered pair $(-3,1)$ means that $x=-3$ and $y=1$. Let us see if either pair is a solution.

a) Checking the ordered pair $(1,-3)$ - this means that $x=1$ and $y=-3$. We substitute these values into both sides of the equation $x^2-y^2=2y-x+3$.

$$\begin{aligned}
\text{the left-hand side is}\quad \text{LHS} &= 1^2-(-3)^2=1-9=-8\\
\text{the right-hand side is}\quad \text{RHS} &= 2(-3)-1+3=-6-1+3=-7+3=-4\\
-8 &\neq -4\\
\text{LHS} &\neq \text{RHS}
\end{aligned}$$

The ordered pair $(1,-3)$ is not a solution of this equation.

b) Checking the ordered pair $(-3,1)$ - this means that $x=-3$ and $y=1$. We substitute these values into both sides of the equation $x^2-y^2=2y-x+3$.

$$\begin{aligned}
\text{the left-hand side is} \quad \text{LHS} &= (-3)^2-1^2=9-1=8 \\
\text{the right-hand side is} \quad \text{RHS} &= 2\cdot 1-(-3)+3=2+3+3=8 \\
8 &= 8 \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

The ordered pair $(-3,1)$ is a solution of this equation.

Definition: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, we found that -5 is a solution of $-x^2+3=4x-2$. As it turns out, -5 is not the only solution. We leave to the reader to verify that 1 is also a solution of the equation. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We will start with the simplest equations, linear equations.

Linear equations are a fundamental concept and tool in mathematics. To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. In what follows, we will assume that the reader knows how to solve one-step and two-step linear equations. If you need to review that, please see the handout Solving One- and Two-Step Linear Equations.

Example 3: Solve each of the given equations. Make sure to check your solutions.

a) $2x-8=5x+10$ b) $7a-12=-a+20$ c) $-4x+2=-x+17$ d) $\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4}$

Solution: Notice that in each equation, the unknown appears on both sides. This will be the first thing we will address.

a)

$$\begin{aligned}
2x-8 &= 5x+10 && \text{subtract } 2x \\
-8 &= 3x+10 && \text{subtract } 10 \\
-18 &= 3x && \text{divide by } 3 \\
-6 &= x
\end{aligned}$$

So the only solution of this equation is -6. We check; if $x=-6$,

$$\text{LHS}=2(-6)-8=-12-8=-20 \text{ and } \text{RHS}=5(-6)+10=-30+10=-20 \implies \text{LHS}=\text{RHS}$$

So our solution, $\boxed{x=-6}$ is correct.

b)

$$\begin{aligned}
7a-12 &= -a+20 && \text{add } a \\
8a-12 &= 20 && \text{add } 12 \\
8a &= 32 && \text{divide by } 8 \\
a &= 4
\end{aligned}$$

So the only solution of this equation is 4. We check; if $a=4$,

$$\text{LHS}=7\cdot 4-12=28-12=16 \text{ and } \text{RHS}=-4+20=16 \implies \text{LHS}=\text{RHS}$$

So our solution, $\boxed{a=4}$ is correct.

c)

$$\begin{aligned} -4x+2 &= -x+17 && \text{add } 4x \\ 2 &= 3x+17 && \text{subtract } 17 \\ -15 &= 3x && \text{divide by } 3 \\ -5 &= x \end{aligned}$$

We check; if $x=-5$, then

$$\text{LHS} = -4(-5)+2 = 20+2 = 22 \ \text{ and } \ \text{RHS} = -(-5)+17 = 5+17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x=-5}$ is correct.

d)

$$\begin{aligned} \frac{1}{2}m-1 &= \frac{5}{4}m-\frac{1}{4} && \text{subtract } \frac{1}{2}m \\ -1 &= \frac{3}{4}m-\frac{1}{4} && \text{add } \frac{1}{4} \\ -\frac{3}{4} &= \frac{3}{4}m && \text{divide by } \frac{3}{4} \\ -1 &= m \end{aligned}$$

margin work: $\frac{5}{4}-\frac{1}{2}=\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$

$-1+\frac{1}{4}=\frac{-4}{4}+\frac{1}{4}=-\frac{3}{4}$

$-\frac{3}{4}\div\frac{3}{4}=-\frac{3}{4}\cdot\frac{4}{3}=-1$

So the only solution of this equation is -1. We check; if $m=-1$,

$$\text{LHS} = \frac{1}{2}(-1)-1 = -\frac{1}{2}-1 = \frac{-1}{2}-\frac{2}{2} = -\frac{3}{2} \ \text{ and}$$

$$\text{RHS} = \frac{5}{4}(-1)-\frac{1}{4} = -\frac{5}{4}-\frac{1}{4} = -\frac{6}{4} = -\frac{3}{2} \qquad \implies \qquad \text{LHS} = \text{RHS}$$

So our solution, $\boxed{m=-1}$ is correct.

Linear equations might be more complicated. Most often we will be dealing with the distributive law. Also, these equations can be classified based on their solution sets. Consider each of the following.

Example 4: Solve each of the given equations. Make sure to check your solutions.

a) $3x-2(4-x)=3(3x-1)-(x-7)$ b) $4(y-2)-6(3y-5)=5-2(7y+1)$ c) $\frac{2}{3}x-4-\frac{1}{6}(x+6)=\frac{1}{2}(x-10)$

Solution: a) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 3x-2(4-x) &= 3(3x-1)-(x-7) && \text{eliminate parentheses} \\ 3x-8+2x &= 9x-3-x+7 && \text{combine like terms} \\ 5x-8 &= 8x+4 && \text{subtract } 5x \\ -8 &= 3x+4 && \text{subtract } 4 \\ -12 &= 3x && \text{divide by } 3 \\ -4 &= x \end{aligned}$$

Caution! $-2(-x)=2x$ and $-(-7)=7$

We check: if $x=-4$, then

$$\begin{aligned} \text{LHS} &= 3(-4)-2(4-(-4)) = 3(-4)-2\cdot 8 = -12-16 = -28 \quad \text{and} \\ \text{RHS} &= 3(3(-4)-1)-(-4-7) = 3(-12-1)-(-11) = 3(-13)+11 = -39+11 = -28 \implies \text{LHS} = \text{RHS} \end{aligned}$$

So our solution, $\boxed{x=-4}$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 4(y-2)-6(3y-5) &= 5-2(7y+1) && \text{eliminate parentheses} \\ 4y-8-18y+30 &= 5-14y-2 && \text{combine like terms} \\ -14y+22 &= -14y+3 && \text{add } 14y \\ 22 &= 3 \end{aligned}$$

Something different happened here. When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $22=3$. No matter what the value of the unknown is, this statement can not be made true. Indeed, our last line is an **unconditionally false statement**. This means that there is no number that could make this statement true, and so this equation **has no solution**. An equation like this is called a **contradiction**.

c) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} \frac{2}{3}x-4-\frac{1}{6}(x+6) &= \frac{1}{2}(x-10) && \text{eliminate parentheses} \\ \frac{2}{3}x-4-\frac{1}{6}x-1 &= \frac{1}{2}x-5 && \text{combine like terms} \quad \text{margin work: } \frac{2}{3}-\frac{1}{6}=\frac{4-1}{6}=\frac{3}{6}=\frac{1}{2} \\ \frac{1}{2}x-5 &= \frac{1}{2}x-5 && \text{subtract } \frac{1}{2}x \\ -5 &= -5 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared again from both sides. We are left with the statement $-5=-5$. No matter what the value of the unknown is, this statement is always true. Indeed, our last line is an **unconditionally true statement**. This means that every number makes make this statement true, and so the solution set of this equation is the set of all numbers. An equation like this is called an **identity**.

We often use identities in mathematics, although it seems at first that we would not need equations whose solution set is every number. Consider the following equation: $a+b=b+a$. This equation is an identity, because every pair of numbers is a solution. We use this identity to express a property of *addition*: that the sum of two numbers does not depend on the order of the two numbers.

Based on their solution sets, these equations can be classified as belonging to one of the following three groups.

1. If the last line is of the form $x=5$, the equation is called **conditional**. (This is because the truth value of the statement depends on the value of x. True if x is 5, false otherwise.) A conditional equation has exactly one solution.

2. If the last line is of the form $1=1$, the equation is unconditionally true. Such an equation is called an **identity** and all numbers are solutions of it.

3. If the last line is of the form $3=14$, the equation is unconditionally false. Such an equation is called a **contradiction** its solution set is the empty set.

Discussion: Classify each of the following equations as conditional, identity, or contradiction.

a) $3x+1=3x-1$ b) $2x-4=7x-4$ c) $x-4=4-x$ d) $x-1=-1+x$

Example 4: Solve each of the given equations. Make sure to check your solutions.

a) $\dfrac{2x-5}{3}-\dfrac{x-2}{5}=x-5$ b) $\dfrac{2}{3}(x-1)-\dfrac{1}{2}\left(x+\dfrac{3}{5}\right)=-x+\dfrac{37}{10}$

Solution: a) The main idea here is that we can clear denominators of fractions in equations by muliplying by a suitable number. As always, we will multiply both sides by a suitable number.

$$\begin{aligned}
\frac{2x-5}{3}-\frac{x-2}{5} &= x-5 && \text{we write everything as a fraction}\\
\frac{2x-5}{3}-\frac{x-2}{5} &= \frac{x-5}{1} && \text{bring all three fractions to the common denominator}\\
\frac{5(2x-5)}{15}-\frac{3(x-2)}{15} &= \frac{15(x-5)}{15} && \text{clear denominators by multiplying by 15}\\
5(2x-5)-3(x-2) &= 15(x-5) && \text{remove parentheses}\\
10x-25-3x+6 &= 15x-75 && \text{combine like terms}\\
7x-19 &= 15x-75 && \text{subtract } 7x\\
-19 &= 8x-75 && \text{add 75}\\
56 &= 8x && \text{divide by 8}\\
7 &= x
\end{aligned}$$

We check: if $x=7$, then

$$\text{LHS}=\frac{2\cdot 7-5}{3}-\frac{7-2}{5}=\frac{14-5}{3}-\frac{5}{5}=\frac{9}{3}-1=3-1=2 \quad \text{and} \quad \text{RHS}=7-5=2$$

and so our solution, $\boxed{x=7}$ is correct.

b) There are several methods available. The method presented here is focusing how similar this equation is to the previous example.

$$\begin{aligned}
\frac{2}{3}(x-1)-\frac{1}{2}\left(x+\frac{3}{5}\right) &= -x+\frac{37}{10} && x=1x=\frac{5}{5}x \quad \text{and} \quad -x=-1x=-\frac{10}{10}x\\
\frac{2}{3}(x-1)-\frac{1}{2}\left(\frac{5x}{5}+\frac{3}{5}\right) &= -\frac{10x}{10}+\frac{37}{10}\\
\frac{2}{3}\cdot\frac{x-1}{1}-\frac{1}{2}\cdot\frac{5x+3}{5} &= \frac{-10x+37}{10}\\
\frac{2(x-1)}{3}-\frac{5x+3}{10} &= \frac{-10x+37}{10} && \text{bring all fractions to the common denominator}\\
\frac{20(x-1)}{30}-\frac{3(5x+3)}{30} &= \frac{3(-10x+37)}{30} && \text{clear denominator by multiplying by 30}\\
20(x-1)-3(5x+3) &= 3(-10x+37) && \text{remove parentheses}\\
20x-20-15x-9 &= -30x+111 && \text{combine like terms}\\
5x-29 &= -30x+111 && \text{add } 30x\\
35x-29 &= 111 && \text{add 29}\\
35x &= 140 && \text{divide by 35}\\
x &= 4
\end{aligned}$$

We check: if $x=4$, then

$$\begin{aligned}\text{LHS} &= \frac{2}{3}(4-1)-\frac{1}{2}\left(4+\frac{3}{5}\right)=\frac{2}{3}\cdot 3-\frac{1}{2}\left(\frac{20}{5}+\frac{3}{5}\right)=2-\frac{1}{2}\cdot\frac{23}{5}=2-\frac{23}{10}=\frac{20}{10}-\frac{23}{10}=\frac{-3}{10}\\ \text{RHS} &= -4+\frac{37}{10}=-\frac{40}{10}+\frac{37}{10}=-\frac{3}{10}\end{aligned}$$

and so our solution, $\boxed{x=4}$ is correct.

Example 5: Solve the given equation. Make sure to check your solutions.

$$(2x-3)^2-(x+1)(3x-5)=11-(x-1)(3-x)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expanded $(x+1)(3x-5)$ and $(x-1)(3-x)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$$\begin{aligned}(2x-3)^2 &= (2x-3)(2x-3)=4x^2-6x-6x+9=4x^2-12x+9\\ (x+1)(3x-5) &= 3x^2-5x+3x-5=3x^2-2x-5\\ (x-1)(3-x) &= 3x-x^2-3+x=-x^2+4x-3\end{aligned}$$

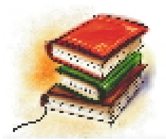
We are now ready to begin solving the equation.

$$\begin{aligned}(2x-3)^2-(x+1)(3x-5) &= 11-(x-1)(3-x) & &\\ 4x^2-12x+9-\left(3x^2-2x-5\right) &= 11-\left(-x^2+4x-3\right) & &\text{to subtract is to add the opposite}\\ 4x^2-12x+9-3x^2+2x+5 &= 11+x^2-4x+3 & &\text{combine like terms}\\ x^2-10x+14 &= x^2-4x+14 & &\text{subtract } x^2\\ -10x+14 &= -4x+14 & &\text{add } 10x\\ 14 &= 6x+14 & &\text{subtract } 14\\ 0 &= 6x & &\text{divide by } 6\\ 0 &= x & &\end{aligned}$$

We check: if $x=0$, then

$$\begin{aligned}\text{LHS} &= (2\cdot 0-3)^2-(0+1)(3\cdot 0-5)=(-3)^2-1(-5)=9+5=14\\ \text{RHS} &= 11-(0-1)(3-0)=11-(-1)3=11+3=14\end{aligned}$$

and so our solution, $\boxed{x=0}$ is correct.



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x+3=4x+9$
2. $3w-5=5(w+1)$
3. $3y-9=-2y+4$
4. $4-x=3(x-7)$
5. $7(j-5)+9=2(-2j+5)+5j$
6. $3(x-5)-5(x-1)=-2x+1$
7. $\dfrac{3-x}{4}-\dfrac{10-3x}{5}=x+2$
8. $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$
9. $\dfrac{2}{3}(x-1)=\dfrac{3}{5}(x-4)+1$
10. $\dfrac{2}{3}(x-7)=\dfrac{4}{5}(x+1)$
11. $\dfrac{x+2}{4}-\dfrac{x-3}{5}=20-x$
12. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$
13. $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$
14. $12-(2p-1)(p+1)=-2(-p+5)^2$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x-3=x+9$
2. $-x+13=2x+1$
3. $-2x+4=5x-10$
4. $5x-7=6x+8$
5. $8x-1=3x+19$
6. $-7x-1=3x-21$
7. $3(x-4)=2(x+5)$
8. $4(5x+1)=6x+4$
9. $a-3=5(a-1)-2$
10. $3y-2=-2y+18$
11. $8(x-3)-3(5-2x)=x$
12. $5(x-1)-3(x+1)=3x-8$
13. $-2x-(3x-1)=2(5-3x)$
14. $3(x-4)+5(x+8)=2(x-1)$
15. $5(x-1)-3(-x+1)=-3+8x$
16. $\dfrac{3x-1}{5}-\dfrac{7-x}{3}=2x+6$
17. $\dfrac{3x-1}{4}+\dfrac{8-4x}{3}=-3-x$
18. $\dfrac{3x-2}{5}+\dfrac{x+4}{3}=\dfrac{14(x+1)}{15}$
19. $\dfrac{3}{8}x+1\dfrac{4}{5}=\dfrac{1}{4}x+1\dfrac{3}{10}$
20. $\dfrac{2x+1}{3}-\dfrac{3-x}{2}=x-2$
21. $\dfrac{2}{3}x-1=-\dfrac{2}{3}\left(x+\dfrac{1}{2}\right)$
22. $2(b+1)-5(b-3)=2(b-7)+1$
23. $3(2x-1)-5(2-x)=4(x-1)+5$
24. $3(2x-7)-2(5x+2)=-5x-30$
25. $3(x-4)-4(x-3)=3(x-2)+2(3-x)$
26. $2x(3x-1)-x(5x-2)=(x-1)^2$
27. $y^2-(y-1)^2+(y-2)^2=(y-3)(y-5)$
28. $(3x)^2-(x+3)(5x-3)=(5-2x)^2-16$
29. $(w+4)(1-2w)=3w-2(w-3)^2$
30. $(2x-3)^2-3(x-2)^2=10-(x-2)(7-x)$
31. $(2-w)^2-(2w-3)^2+7=(w-2)(5-3w)$
32. $3(a+11)-a(8-3a)=3(a-2)^2$
33. $-5(2x-1)-(4-x)^2=3-(x+1)^2$
34. $5(-3-x)-3x(x-2)=x-3(x+2)(x-5)$
35. $2(-m-2)^2-(m-2)^2=8m+(m+2)^2$
36. $(3a-5)(2-a)-(2a-1)(a+3)=-5a^2-7$
37. $\dfrac{1}{2}(x-3)^2-\dfrac{1}{2}(x+1)^2=4(x-7)$

Answers

Discussion

a) contradiction b) conditional c) conditional d) identity

Sample Problems

1. -3 2. -5 3. $\frac{13}{5}$ 4. $\frac{25}{4}$ 5. 6 6. no solution 7. -5 8. identity, all numbers are solution 9. -11
10. -41 11. 18 12. 2 13. 0 14. 3

Practice Problems

1. 3 2. 4 3. 2 4. -15 5. 4 6. 2 7. 22 8. 0 9. 1 10. 4 11. 3 12. 0 13. 9 14. -5
15. there is no solution 16. -8 17. -13 18. all real numbers are solution 19. -4 20. -5 21. $\frac{1}{2}$ 22. 6
23. 2 24. -5 25. 0 26. $\frac{1}{2}$ 27. 2 28. 0 29. 1 30. 3 31. 4 32. -3 33. there is no solution
34. -5 35. all real numbers are solution 36. 0 37. 4

Sample Problems Solutions

1. $2x+3=4x+9$
   Solution:

$$\begin{aligned} 2x+3 &= 4x+9 && \text{subtract } 2x \text{ from both sides} \\ 3 &= 2x+9 && \text{subtract } 9 \text{ from both sides} \\ -6 &= 2x && \text{divide both sides by 2} \\ -3 &= x \end{aligned}$$

   We check: if $x=-3$, then

$$\begin{aligned} \text{LHS} &= 2(-3)+3=-6+3=-3 \\ \text{RHS} &= 4(-3)+9=-12+9=-3 \end{aligned}$$

   Thus our solution, $\boxed{x=-3}$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w-5=5(w+1)$
   Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 3w-5 &= 5(w+1) \\ 3w-5 &= 5w+5 && \text{subtract } 3w \text{ from both sides} \\ -5 &= 2w+5 && \text{subtract } 5 \text{ from both sides} \\ -10 &= 2w && \text{divide both sides by 2} \\ -5 &= w \end{aligned}$$

   We check. If $w=-5$, then

$$\begin{aligned} \text{LHS} &= 3(-5)-5=-15-5=-20 \\ \text{RHS} &= 5((-5)+1)=5(-4)=-20 \end{aligned}$$

   Thus our solution, $\boxed{w=-5}$ is correct.

3. $3y-9=-2y+4$
   Solution:

$$\begin{aligned} 3y-9 &= -2y+4 && \text{add } 2y \text{ to both sides} \\ 5y-9 &= 4 && \text{add } 9 \text{ to both sides} \\ 5y &= 13 && \text{divide both sides by } 5 \\ y &= \frac{13}{5} \end{aligned}$$

   We check. If $y=\frac{13}{5}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{13}{5}\right)-9=\frac{3}{1}\cdot\frac{13}{5}-9=\frac{39}{5}-\frac{9}{1}=\frac{39}{5}-\frac{45}{5}=\frac{-6}{5}=-\frac{6}{5} \\ \text{RHS} &= -2\left(\frac{13}{5}\right)+4=\frac{-2}{1}\cdot\frac{13}{5}+\frac{4}{1}=\frac{-26}{5}+\frac{20}{5}=\frac{-6}{5}=-\frac{6}{5} \end{aligned}$$

   Thus $\boxed{y=\frac{13}{5}}$ is the correct solution.

4. $4-x=3(x-7)$
Solution: We first apply the law of distributivity to simplify the right-hand side.

$$\begin{array}{rcll} 4-x &=& 3(x-7) & \text{distribute } 3 \\ 4-x &=& 3x-21 & \text{add } x \text{ to both sides} \\ 4 &=& 4x-21 & \text{add } 21 \text{ to both sides} \\ 25 &=& 4x & \text{divide both sides by } 4 \\ \dfrac{25}{4} &=& x & \end{array}$$

We check. If $x=\dfrac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 4-x=4-\frac{25}{4}=\frac{4}{1}-\frac{25}{4}=\frac{16}{4}-\frac{25}{4}=\frac{16-25}{4}=\frac{-9}{4}=-\frac{9}{4} \\ \text{RHS} &= 3(x-7)=3\left(\frac{25}{4}-7\right)=3\left(\frac{25}{4}-\frac{7}{1}\right)=3\left(\frac{25}{4}-\frac{28}{4}\right)=3\left(\frac{25-28}{4}\right) \\ &= 3\left(\frac{-3}{4}\right)=\frac{3}{1}\cdot\frac{-3}{4}=\frac{-9}{4}=-\frac{9}{4} \end{aligned}$$

Thus our solution, $\boxed{x=\dfrac{25}{4}}$ is correct.

5. $7(j-5)+9=2(-2j+5)+5j$
Solution:

$$\begin{array}{rcll} 7(j-5)+9 &=& 2(-2j+5)+5j & \text{distribute on both sides} \\ 7j-35+9 &=& -4j+10+5j & \text{combine like terms} \\ 7j-26 &=& j+10 & \text{subtract } j \\ 6j-26 &=& 10 & \text{add } 26 \\ 6j &=& 36 & \text{divide by } 6 \\ j &=& 6 & \end{array}$$

We check: if $j=6$, then

$$\begin{aligned} \text{LHS} &= 7(6-5)+9=7\cdot 1+9=7+9=16 \\ \text{RHS} &= 2(-2\cdot 6+5)+5\cdot 6=2(-12+5)+30=2(-7)+30=-14+30=16 \end{aligned}$$

Thus our solution, $\boxed{j=6}$ is correct.

6. $3(x-5)-5(x-1)=-2x+1$
Solution:

$$\begin{array}{rcll} 3(x-5)-5(x-1) &=& -2x+1 & \text{multiply out parentheses} \\ 3x-15-5x+5 &=& -2x+1 & \text{combine like terms} \\ -2x-10 &=& -2x+1 & \text{add } 2x \\ -10 &=& 1 & \end{array}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, $\boxed{\text{there is no solution}}$ for this equation. This type of an equation is called a **contradiction**.

7. $\dfrac{3-x}{4}-\dfrac{10-3x}{5}=x+2$

Solution:

$$\begin{aligned}
\frac{3-x}{4}-\frac{10-3x}{5} &= x+2 && \text{make everything a fraction}\\
\frac{3-x}{4}-\frac{10-3x}{5} &= \frac{x+2}{1} && \text{common denominator}\\
\frac{5(3-x)}{20}-\frac{4(10-3x)}{20} &= \frac{20(x+2)}{20} && \text{multiply by 20}\\
5(3-x)-4(10-3x) &= 20(x+2) && \text{distribute}\\
15-5x-40+12x &= 20x+40 && \text{combine like terms}\\
7x-25 &= 20x+40 && \text{subtract } 7x\\
-25 &= 13x+40 && \text{subtract } 40\\
-65 &= 13x && \text{divide by 13}\\
-5 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{3-(-5)}{4}-\frac{10-3(-5)}{5}=\frac{8}{4}-\frac{25}{5}=2-5=-3\\
\text{RHS} &= -5+2=-3
\end{aligned}$$

Thus our solution, $\boxed{x=-5}$ is correct.

8. $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$

Solution:

$$\begin{aligned}
\frac{3x+17}{2} &= x-1+\frac{x+19}{2} && \text{express everything as a fraction}\\
\frac{3x+17}{2} &= \frac{x-1}{1}+\frac{x+19}{2} && \text{bring everything to the common denominator}\\
\frac{3x+17}{2} &= \frac{2(x-1)}{2}+\frac{x+19}{2} && \text{add fractions on right hand side}
\end{aligned}$$

$$\begin{aligned}
\frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses}\\
\frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms}\\
\frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2}\\
3x+17 &= 3x+17
\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and $\boxed{\text{all real numbers}}$ are solution.

9. $\dfrac{2}{3}(x-1) = \dfrac{3}{5}(x-4)+1$

Solution: We re-write the expressions as fractions.

$$\begin{aligned}
\frac{2(x-1)}{3} &= \frac{3(x-4)}{5}+\frac{1}{1} && \text{common denominator is 15}\\
\frac{5\cdot 2(x-1)}{5\cdot 3} &= \frac{3\cdot 3(x-4)}{3\cdot 5}+\frac{15}{15}\\
\frac{10(x-1)}{15} &= \frac{9(x-4)}{15}+\frac{15}{15} && \text{multiply by 15}\\
10(x-1) &= 9(x-4)+15 && \text{distribute}\\
10x-10 &= 9x-36+15 && \text{combine like terms}\\
10x-10 &= 9x-21 && \text{subtract } 9x\\
x-10 &= -21 && \text{add 10}\\
x &= -11
\end{aligned}$$

We check. If $x=-11$, then

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-11-1) = \frac{2}{3}(-12) = -8\\
\text{RHS} &= \frac{3}{5}(-11-4)+1 = \frac{3}{5}(-15)+1 = -9+1 = -8
\end{aligned}$$

Thus our solution, $\boxed{x=-11}$ is correct.

10. $\dfrac{2}{3}(x-7) = \dfrac{4}{5}(x+1)$

Solution:

$$\begin{aligned}
\frac{2}{3}(x-7) &= \frac{4}{5}(x+1)\\
\frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1}\\
\frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator}\\
\frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} && \text{multiply both sides by 15}
\end{aligned}$$

$$\begin{aligned}
10(x-7) &= 12(x+1) && \text{multiplty out parentheses}\\
10x-70 &= 12x+12 && \text{subtract } 10x\\
-70 &= 2x+12 && \text{subtract 12}\\
-82 &= 2x && \text{divide by 2}\\
-41 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-41-7) = \frac{2}{3}(-48) = -32\\
\text{RHS} &= \frac{4}{5}(-41+1) = \frac{4}{5}(-40) = -32
\end{aligned}$$

Thus our solution, $\boxed{x=-41}$ is correct.

11. $\dfrac{x+2}{4}-\dfrac{x-3}{5}=20-x$
Solution:

$$\begin{aligned}
\frac{x+2}{4}-\frac{x-3}{5} &= 20-x && \text{make everything a fraction}\\
\frac{x+2}{4}-\frac{x-3}{5} &= \frac{20-x}{1} && \text{common denominator is 20}\\
\frac{5(x+2)}{20}-\frac{4(x-3)}{20} &= \frac{20(20-x)}{20} && \text{multiply by 20}\\
5(x+2)-4(x-3) &= 20(20-x) && \text{distribute}\\
5x+10-4x+12 &= 400-20x && \text{combine like terms}\\
x+22 &= -20x+400 && \text{add } 20x\\
21x+22 &= 400 && \text{subtract 22}\\
21x &= 378 && \text{divide by 21}\\
x &= 18
\end{aligned}$$

We check. If $x=18$, then

$$\begin{aligned}
\text{LHS} &= \frac{18+2}{4}-\frac{18-3}{5}=\frac{20}{4}-\frac{15}{5}=5-3=2\\
\text{RHS} &= 20-18=2
\end{aligned}$$

Thus $\boxed{x=18}$ is indeed the solution.

12. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x-3)^2-(2x-5)(x+1) &= 5-(x-1)^2\\
x^2-3x-3x+9-\left(2x^2+2x-5x-5\right) &= 5-\left(x^2-x-x+1\right) && \text{combine like terms}\\
x^2-6x+9-\left(2x^2-3x-5\right) &= 5-\left(x^2-2x+1\right) && \text{distribute}\\
x^2-6x+9-2x^2+3x+5 &= 5-x^2+2x-1 && \text{combine like terms}\\
-x^2-3x+14 &= -x^2+2x+4 && \text{add } x^2\\
-3x+14 &= 2x+4 && \text{add } 3x\\
14 &= 5x+4 && \text{subtract 4}\\
10 &= 5x && \text{divide by 5}\\
2 &= x
\end{aligned}$$

We check. If $x=2$, then

$$\begin{aligned}
\text{LHS} &= (2-3)^2-(2\cdot 2-5)(2+1)=(-1)^2-(4-5)(2+1)=(-1)^2-(-1)\cdot 3\\
&= 1-(-3)=4\\
\text{RHS} &= 5-(2-1)^2=5-1^2=5-1=4
\end{aligned}$$

Thus $\boxed{x=2}$ is indeed the solution.

13. $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x+1)^2-(2x-1)^2+(3x)^2 &= 6x(x-2) \\
x^2+x+x+1-\left(4x^2-2x-2x+1\right)+9x^2 &= 6x^2-12x \\
x^2+2x+1-\left(4x^2-4x+1\right)+9x^2 &= 6x^2-12x && \text{distribute} \\
x^2+2x+1-4x^2+4x-1+9x^2 &= 6x^2-12x && \text{combine like terms} \\
6x^2+6x &= 6x^2-12x && \text{subtract } 6x^2 \\
6x &= -12x && \text{add } 12x \\
18x &= 0 && \text{divide by } 18 \\
x &= 0
\end{aligned}$$

We check. If $x=0$, then

$$\begin{aligned}
\text{LHS} &= (0+1)^2-(2\cdot 0-1)^2+(3\cdot 0)^2 = 1^2-(-1)^2+(0)^2 \\
&= 1-1+0=0 \\
\text{RHS} &= 6\cdot 0\cdot(0-2) = 6\cdot 0\cdot(-2)=0
\end{aligned}$$

Thus $\boxed{x=0}$ is indeed the solution.

14. $12-(2p-1)(p+1)=-2(-p+5)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
12-(2p-1)(p+1) &= -2(-p+5)^2 \\
12-\left(2p^2+2p-p-1\right) &= -2\left(p^2-5p-5p+25\right) && \text{combine like terms} \\
12-\left(2p^2+p-1\right) &= -2\left(p^2-10p+25\right) && \text{distribute} \\
12-2p^2-p+1 &= -2p^2+20p-50 && \text{combine like terms} \\
-2p^2-p+13 &= -2p^2+20p-50 && \text{add } 2p^2 \\
-p+13 &= 20p-50 && \text{add } p \\
13 &= 21p-50 && \text{add } 50 \\
63 &= 21p && \text{divide by } 21 \\
3 &= p
\end{aligned}$$

We check. If $p=3$, then

$$\begin{aligned}
\text{LHS} &= 12-(2\cdot 3-1)(3+1) = 12-(6-1)(3+1) = 12-5\cdot 4 = 12-20 = -8 \\
\text{RHS} &= -2(-3+5)^2 = -2\cdot 2^2 = -2\cdot 4 = -8
\end{aligned}$$

Thus $\boxed{p=3}$ is indeed the solution.