Definition: The Fibonacci sequence starts with 1 and 1 and for all other terms in the sequence, we must add the last two terms.

$$F_1 = 1 \qquad F_2 = 2 \qquad \text{and for all } n \geq 1, \quad F_n + F_{n+1} = F_{n+2}$$

So the first few terms of the Fibonacci sequence are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

The definition shown above is a **recursive** one. If we are needed to compute the 100th term of the sequence, we would be forced to compute first the first 99 terms in the sequence. So we are naturally interested in finding a formula that enables us to compute the 100th element directly. Such a formula is callled **explicit**. Like so many things about this sequence, the explicit formula for its nth term is fascinating and surprising. We will derive this formula later.

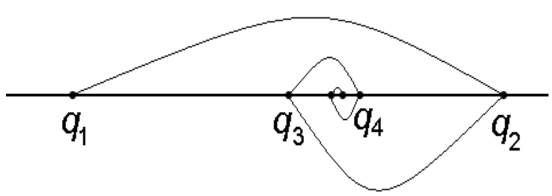
The Fibonacci sequence is named after Leonardo Fibonacci and has very strange and beautiful properties. A lot of these properties are connected to the golden mean, $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.618\,033\,988\,749\,9$.

Consider now another sequence, $\{q_n\}$ that is formed by taking the quotients of consecutive term in the Fibonacci sequence. That is, $\dfrac{1}{1}, \dfrac{2}{1}, \dfrac{3}{2}, \dfrac{5}{3}, \dfrac{8}{5}, \dfrac{13}{8}, \ldots..$

$$q_n = \frac{F_n}{F_{n+1}} \quad \text{for all natural number } n.$$

The decimal presentations of the terms in this sequence show an interesting pattern.

$$\frac{1}{1} = 1 \qquad \frac{2}{1} = 2 \qquad \frac{3}{2} = 1.5 \qquad \frac{5}{3} \approx 1.66667 \qquad \frac{8}{5} = 1.6 \qquad \frac{13}{8} = 1.625 \qquad \frac{21}{13} \approx 1.615\,384\,6 \qquad \frac{34}{21} \approx 1.619\,047\,62$$



The quotients oscillate back and forth and seem to be closer and closer to each other. Amazingly, there is only one number that is inside all of the "swirls" shown on the picture above. These ratios approach a single number. We call this number the limit of this sequence and we compute its exact value in the sample problems.

Definition: A Fibonacci-type of a sequence starts with any two real numbers and the rest of the sequence is generated the same way the Fibonacci sequence is.

$$f_1, f_2 \in \mathbb{R} \text{ and for all } n \geq 1, \quad f_n + f_{n+1} = f_{n+2}$$

Suppose we start with $f_1 = 3$ and $f_2 = 4$. The first few terms of this Fibonacci-type sequence are

$$3,\ 4,\ 7,\ 11,\ 15,\ 26,\ 41,\ 67,\ 108,\ 175,\ \ldots$$

We can define operations on Fibonacci-type sequences. Consider $\{a_n\}$ and $\{b_n\}$ defined as follows:

$$\{a_n\} \ : \quad 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ \ldots$$
$$\{b_n\} \ : \quad 2,\ 7,\ 9,\ 16,\ 25,\ 41,\ 66,\ 107,\ 173,\ 280,\ \ldots$$

We can multiply a sequence by a number by multiplying each term by that number:

$$2\{a_n\}: \quad 6,\ 8,\ 14,\ 22,\ 30,\ 52,\ 82,\ 134,\ 216,\ 350,\ \ldots$$

and the resulting sequence is still Fibonacci-type.

We can also add two sequences by adding them term by term:

$$c_n = a_n + b_n$$

$$c_n = \{a_n + b_n\}: \quad 5,\ 11,\ 16,\ 27,\ 43,\ 70,\ 113,\ 183,\ \ldots$$

and the sum is again Fibonacci-type. Also, it is very easy to see that every Fibonacci-type sequence is uniquely determined by its first two terms.

These properties are used when we derive the explicit formula for the nth term of the Fibonacci sequence.

Sample Problems

1. Solve the equation $x^2 = x + 1$.

2. Define $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$. Prove each of the following.

   a) $1 - \varphi = \psi$ b) $-\dfrac{1}{\varphi} = \psi$

3. Find the limit of of the sequence formed from consecutive terms in the Fibonacci sequence. In short, compute $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n}$.

4. Definition: Two positive integers are relatively prime if their greatest common divisor is 1.

   Prove that any two consecutive terms of the Fibonacci sequence are relatively prime.

5. Consider a Fibonacci-type of a sequence with first term 1 and second term x. Is there a value of x for which all terms of the sequence fall between -100 and 100?

6. Definition: A geometric sequence is defined as $a, ar, ar^2, ar^3, ar^4, \ldots\ldots$. The number r is called the common ratio of the sequence because if $r \neq 0$, then $r = \dfrac{a_{n+1}}{a_n}$ for all $n \in \mathbb{N}$. It is clear that a geometric sequence is determined by its first element and common ratio. One great advantage of a geometric sequence over a Fibonacci-type of a sequence is that there is a very easy explicit formula for the nth term of the sequence: $a_n = ar^{n-1}$.

   Is there a Fibonacci-type sequence that is also a geometric series?

7. Consider the geometric sequence defined by first element 1 and common ratio $r = \dfrac{1+\sqrt{5}}{2}$. Compute the exact value of the 9th term in the sequence.

8. Use results form the previous problems to find the explicit formula for the nth term of the Fibonacci sequence.

Solutions - Sample Problems

1. Solve the equation $x^2 = x + 1$

   Solution 1. Using the quadratic formula

$$x^2 - x - 1 = 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1-(-4)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

   Solution 2. Completing the square

$$\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4}$$

$$\begin{aligned} x^2 - x - 1 &= 0 \\ \underbrace{x^2 - x + \frac{1}{4}} - \frac{1}{4} - 1 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \frac{5}{4} &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 &= 0 \\ \left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right) &= 0 \end{aligned}$$

$$x_1 = \frac{1}{2} - \frac{\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} \qquad x_2 = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

2. Define $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$. Prove each of the following.

   a) $1 - \varphi = \psi$

   Solution:

$$1 - \varphi = 1 - \frac{1+\sqrt{5}}{2} = \frac{2}{2} - \frac{1+\sqrt{5}}{2} = \frac{2-\left(1+\sqrt{5}\right)}{2} = \frac{2-1-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \psi$$

   b) $-\dfrac{1}{\varphi} = \psi$

   Solution: We rationalize the radical expression $-\dfrac{1}{\varphi}$ using its conjugate.

$$\begin{aligned} -\frac{1}{\varphi} &= -\frac{2}{\sqrt{5}+1} = -\frac{2}{\sqrt{5}+1}\cdot 1 = -\frac{2}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1} = -\frac{2\left(\sqrt{5}-1\right)}{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)} = -\frac{2\left(\sqrt{5}-1\right)}{4} \\ &= -\frac{\sqrt{5}-1}{2} = \frac{1-\sqrt{5}}{2} = \psi \end{aligned}$$

3. Find the limit of of the sequence formed from consecutive terms in the Fibonacci sequence. In short, compute $\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}$.

Solution 1: Let us assume first that the consecutive terms

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13},\ \frac{34}{21},\ \frac{55}{34},\ \frac{89}{55},\ \frac{144}{89},\ \ldots$$

do approach a single number x. Imagine we are much further into the sequence, after millions and millions of terms. Then these numbers are very close to x and thus also very close to each other. Sort of like

$$\begin{aligned}
\frac{89}{55} &\approx \frac{144}{89} \approx x && \text{we are in the Fibonacci sequence: } 144 = 89 + 55 \\
\frac{89}{55} &\approx \frac{89+55}{89} \\
\frac{89}{55} &\approx \frac{89}{89} + \frac{55}{89} \\
\frac{89}{55} &\approx 1 + \frac{55}{89} \\
x &\approx 1 + \frac{1}{x}
\end{aligned}$$

Using more general notation, we arrive to the same conclusion. For very large values of n,

$$\begin{aligned}
\frac{F_{n+1}}{F_n} &\approx \frac{F_{n+2}}{F_{n+1}} \approx x && \text{we are in the Fibonacci sequence: } F_{n+2} = F_n + F_{n+1} \\
\frac{F_{n+1}}{F_n} &\approx \frac{F_n + F_{n+1}}{F_{n+1}} \\
\frac{F_{n+1}}{F_n} &\approx \frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+1}} \\
\frac{F_{n+1}}{F_n} &\approx \frac{F_n}{F_{n+1}} + 1 \\
x &\approx \frac{1}{x} + 1
\end{aligned}$$

We solve the equation $x = 1 + \frac{1}{x}$

$$\begin{aligned}
x &= \frac{1}{x} + 1 && \text{multiply by } x \\
x^2 &= 1 + x \\
x^2 - x - 1 &= 0 \\
x_{1,2} &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

We rule out $\frac{1-\sqrt{5}}{2}$ because it is negative, and the sequence clearly approaches a number above 0.6. The other solution, $\frac{1+\sqrt{5}}{2}$, the golden mean is the limit.

Solution 2: This is the same computation but this time it is presented with calculus notation.

Let us denote $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}$ by x.

$$\begin{aligned}
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\frac{F_{n+2}}{F_{n+1}}\\
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\frac{F_n+F_{n+1}}{F_{n+1}}\\
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}}+\frac{F_{n+1}}{F_{n+1}}\right)\\
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}}+1\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\frac{F_n}{F_{n+1}}+1\\
x &= \frac{1}{x}+1\\
x^2 &= 1+x\\
x^2-x-1 &= 0\\
x_{1,2} &= \frac{1\pm\sqrt{5}}{2}
\end{aligned}$$

We rule out $\frac{1-\sqrt{5}}{2}$ because it is negative, and the sequence clearly approaches a number above 0.6. The other solution, the golden mean is the limit. $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$.

4. Definition: Two positive integers are relatively prime if their greatest common divisor is 1.

   Prove that any two consecutive terms of the Fibonacci sequence are relatively prime.

   Solution: this is an interesting application of proofs by contradiction.

   Suppose for a contradiction that there exist two consecutive terms F_k and F_{k+1} (for some natural number k) that are not relatively prime. Then there exists a positive integer $d>1$ such that d is a divisor of both F_k and F_{k+1}. Then there exist α and β positive integers such that $F_k=\alpha d$ and $F_{k+1}=\beta d$. We claim that then d is also a divisor of F_{k-1}.

$$\begin{aligned}
F_k+F_{k-1} &= F_{k+1}\\
F_{k-1} &= F_{k+1}-F_k=\beta d-\alpha d=d\left(\beta-\alpha\right)
\end{aligned}$$

   Thus d also divides F_{k-1}. Next we similarly prove that d is then also a divisor of F_{k-2} and F_{k-3}, and so on, all the way till F_1. Thus d is a divisor of F_1. This is impossible because $d>1$ and $F_1=1$. This is a contradiction completing our proof.

5. Consider a Fibonacci-type of a sequence with first term 1 and second term x. Is there a value of x for which all terms of the sequence fall between -100 and 100?

   Solution: The Fibonacci sequence vey quickly becomes very large. The question is: how can we ensure that the terms of the sequence do not become large? Consider a Fibonacci-type sequence with first term 1 and second term x.

$$1,\ x,\ x+1,\ 2x+1,\ 3x+2,\ 5x+3,\ 8x+5,\ 13x+8,\ 21x+13,\ \ldots .$$

   eventually the terms are $F_n+F_{n-1}x$. If x is positive, even if tiny, the other part alone, F_n will ensure that the nth term is very large. Thus, if we want the terms to stay small, we need x to be negative.

   This idea generalizes. We want every second term positive and every other term negative, because two consecutive terms with the same sign guarantee that the terms after that get very large. Suppose that a and b are two consecutive terms with the same sign. Then from then on, we have that

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ 5a+8b,\ \ldots,\ F_na+F_{n+1}b\ ,\ \ldots .$$

   So we want alternating sings in the sequence

$$1,\ x,\ x+1,\ 2x+1,\ 3x+2,\ 5x+3,\ 8x+5,\ 13x+8,\ 21x+13,\ \ldots .$$

   That is:

1 positive,	$2x+1$ negative,	$8x+5$ positive,
x negative,	$3x+2$ positive,	$13x+8$ negative,
$x+1$ positive,	$5x+3$ negative,	$21x+13$ positive,

and so on. We solve all these inequalities:

$x<0$

$x+1>0 \implies x>-1$

$2x+1<0 \implies x<-\dfrac{1}{2}$

$3x+2>0 \implies x>-\dfrac{2}{3}$

$5x+3<0 \implies x<-\dfrac{3}{5}$

$8x+5>0 \implies x>-\dfrac{5}{8}$

$13x+8<0 \implies x<-\dfrac{8}{13}$

$21x+13>0 \implies x>-\dfrac{13}{21}$

It looks like x must be between the values defined by consecutive terms of the Fibonacci sequence. These rations display a strange behavior, they spiral around over a smaller and smaller interval (see problem 3). The only difference here is that we are looking at $-\dfrac{F_n}{F_{n+1}}$ instead of $\dfrac{F_{n+1}}{F_n}$. The ratios in this problem approach the negative reciprocal of the golden mean. We rationalize $-\dfrac{2}{\sqrt{5}+1}$ and obtain $\dfrac{1-\sqrt{5}}{2}$. This is the only number that will work for x. This sequence will have terms with alternating signs and thus each term will have a smaller absolute value than the previous term. It is an amazing thought that for any other values of x, the sequence will reach huge numbers and outgrow any bound.

We can present a bit more formal computation: denote the sequence by a_n.

$$\{a_n\}: \quad 1,\ x,\ x+1,\ 2x+1,\ 3x+2,\ 5x+3,\ 8x+5,\ 13x+8,\ 21x+13,\ \ldots.$$

Notice that for all $n\geq 3$

$$a_n = F_{n-1}x + F_{n-2}$$

where $\{F_n\}$ is the Fibonacci sequence.

$$\{F_n\}: \quad 1,\ 1,\ 2,3,5,8,13,21,34,55,89,144$$

We want a_1, a_3, a_5,... positive and a_2, a_4, a_6,... negative. For all n, we need $a_{2n+1}>0$ and $a_{2n}<0$.

$$a_{2n} = F_{2n-1}x + F_{2n-2} \quad \text{and} \quad a_{2n+1} = F_{2n}x + F_{2n-1}$$

$$\begin{aligned} F_{2n-1}x + F_{2n-2} &> 0 \quad \text{and} \quad F_{2n}x + F_{2n-1} < 0 \\ x &> -\frac{F_{2n-2}}{F_{2n-1}} \qquad \text{and} \qquad x < -\frac{F_{2n-1}}{F_{2n}} \quad \text{for all } n \\ -\frac{F_{2n-2}}{F_{2n-1}} &< x < -\frac{F_{2n-1}}{F_{2n}} \quad \text{for all } n \end{aligned}$$

Since these quotiens oscillate around and enclose only a single number, x must be that number. Thus

$$x = \lim_{n\to\infty} -\frac{F_{2n-1}}{F_{2n}} = -\frac{1}{\lim\limits_{m\to\infty}\dfrac{F_{m+1}}{F_m}} = -\frac{1}{\dfrac{1+\sqrt{5}}{2}} = \frac{1-\sqrt{5}}{2}$$

6. Definition: A geometric sequence is defined as $a, ar, ar^2, ar^3, ar^4, \ldots\ldots$ The number r is called the common ratio of the sequence because if $r \neq 0$, then $r = \dfrac{a_{n+1}}{a_n}$ for all $n \in \mathbb{N}$. It is clear that a geometric sequence is determined by its first element and common ratio. One great advantage of a geometric sequence over a Fibonacci-type of a sequence is that there is a very easy explicit formula for the nth term of the sequence: $a_n = ar^{n-1}$.

   Is there a Fibonacci-type sequence that is also a geometric series?

   Solution: Let $\{a_n\}$ be a Fibonacci-type geometric sequence with first element a and comon ratio r. Then the first three elements (since geometric) are

$$a,\ ar,\ ar^2$$

   Let us assume that $a \neq 0$. (The constant zero sequence is both Fibonacci-type and geometric, but not very interesting.) The sequence is also Fibonacci-type and so

$$\begin{aligned} a + ar &= ar^2 \\ 0 &= ar^2 - ar - a \qquad \text{factor out } a \\ 0 &= a\left(r^2 - r - 1\right) \qquad \text{divide by } a \\ 0 &= r^2 - r - 1 \\ r_{1,2} &= \frac{1 \pm \sqrt{1^2 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

   At this point, we are not surprised that we again bumped into the golden mean. Both solutions work, which will be very useful later on.

   Suppose that $a = 1$. Then one sequence is

$$1,\ \frac{1+\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 2+\sqrt{5},\ \frac{7+3\sqrt{5}}{2},\ \ldots$$

   is an increasing sequence that grows unbounded. The other sequence is

$$1,\ \frac{1-\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2},\ 2-\sqrt{5},\ \frac{7-3\sqrt{5}}{2},\ \ldots$$

   is a sequence with alternating sings and thus small terms, and the two sequences appear to be conjugates of each other, term by term.

   The computation above shows that only $r = \dfrac{1 \pm \sqrt{5}}{2}$ will result in a non-zero Fibonacci-type sequence. On the other hand, all other such sequences are just constant muliples of these two. All Fibonacci-type sequences with first term a are of the form

$$a,\ a\left(\frac{1+\sqrt{5}}{2}\right),\ a\left(\frac{3+\sqrt{5}}{2}\right),\ a\left(2+\sqrt{5}\right),\ a\left(\frac{7+3\sqrt{5}}{2}\right),\ \ldots \text{ and}$$

$$a,\ a\left(\frac{1-\sqrt{5}}{2}\right),\ a\left(\frac{3-\sqrt{5}}{2}\right),\ a\left(2-\sqrt{5}\right),\ a\left(\frac{7-3\sqrt{5}}{2}\right),\ \ldots$$

   What makes these sequences special is that their nth term can be so easily determined because they are geometric sequences as well.

7. Consider the geometric sequence defined by first element 1 and common ratio $r = \dfrac{1+\sqrt{5}}{2}$. Compute the exact value of the 9th term in the sequence.

   Solution:

$$a_9 = ar^8 = 1\left(\frac{1+\sqrt{5}}{2}\right)^8 = \left(\frac{1+\sqrt{5}}{2}\right)^8$$

We start with $\left(\frac{1+\sqrt{5}}{2}\right)^2$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{\left(1+\sqrt{5}\right)^2}{2^2} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

We square this number:

$$\left(\frac{1+\sqrt{5}}{2}\right)^4 = \left[\left(\frac{1+\sqrt{5}}{2}\right)^2\right]^2 = \left(\frac{3+\sqrt{5}}{2}\right)^2 = \frac{\left(3+\sqrt{5}\right)^2}{2^2} = \frac{14+6\sqrt{5}}{4} = \frac{7+3\sqrt{5}}{2}$$

We square again:

$$\begin{aligned}\left(\frac{1+\sqrt{5}}{2}\right)^8 &= \left[\left(\frac{1+\sqrt{5}}{2}\right)^4\right]^2 = \left(\frac{7+3\sqrt{5}}{2}\right)^2 = \frac{\left(7+3\sqrt{5}\right)^2}{2^2} = \frac{49+45+42\sqrt{5}}{4} = \frac{49+45+42\sqrt{5}}{4} \\ &= \frac{94+42\sqrt{5}}{4} = \frac{47+21\sqrt{5}}{2}\end{aligned}$$

and so $a_9 = \dfrac{47+21\sqrt{5}}{2}$.

This might seem laborous but if n is large, it is still much better than having to compute all previous terms.

8. Use results form the previous problems to find the explicit formula for the nth term of the Fibonacci sequence.

Solution: We will express the Fibonacci sequence as the sum of two Fibonacci-type geometric sequences.

First, we need to verify that the constant mutplies and sums of Fibonacci-type sequences are still Fibonacci-type. Second, we will use the fact that the first two elements uniquely determine any Fibonaccy-type of sequence.

Define $\{a_n\}$ and $\{b_n\}$ geometric sequences as follows. $a_1 = 1$ and $r_a = \dfrac{1+\sqrt{5}}{2}$ and $b_n = 1$ and $r_b = \dfrac{1-\sqrt{5}}{2}$. Thus

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \quad \text{and} \quad b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

These sequences are also Fibonacci-type (see problem 6). Thus, any constant multiples and sums formed from these sequences will still be Fibonacci-type. Could we use $\{a_n\}$ and $\{b_n\}$ to "concoct" the Fibonacci sequence?

Let x and y be real numbers such that for all n

$$c_n = xa_n + ya_n \text{ and } c_1 = 1 \text{ and } c_2 = 1$$

If we could find such x and y, we would be done because a Fibonacci-type sequence that begins with 1 and 1 is THE fibonacci-sequence.

$$\begin{aligned} 1 &= c_1 = xa_1 + ya_1 \quad \Longrightarrow \quad 1 = x\left(\frac{1+\sqrt{5}}{2}\right) + y\left(\frac{1-\sqrt{5}}{2}\right) \\ 1 &= c_2 = xa_2 + ya_2 \quad \Longrightarrow \quad 1 = x\left(\frac{1+\sqrt{5}}{2}\right)^2 + y\left(\frac{1-\sqrt{5}}{2}\right)^2 \end{aligned}$$

The system $\begin{cases} x\left(\dfrac{1+\sqrt{5}}{2}\right)+y\left(\dfrac{1-\sqrt{5}}{2}\right)=1 \\ x\left(\dfrac{1+\sqrt{5}}{2}\right)^2+y\left(\dfrac{1-\sqrt{5}}{2}\right)^2=1 \end{cases}$ is a linear system in x and y so we should be able to solve it. We simplify both equations. Since

$$\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{\left(1+\sqrt{5}\right)^2}{2^2}=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$$

the system is

$$\begin{cases} x\left(\dfrac{1+\sqrt{5}}{2}\right)+y\left(\dfrac{1-\sqrt{5}}{2}\right)=1 \\ x\left(\dfrac{3+\sqrt{5}}{2}\right)+y\left(\dfrac{3-\sqrt{5}}{2}\right)=1 \end{cases}$$

Let us multiply by 2

$$\begin{cases} x\left(1+\sqrt{5}\right)+y\left(1-\sqrt{5}\right)=2 \\ x\left(3+\sqrt{5}\right)+y\left(3-\sqrt{5}\right)=2 \end{cases}$$

We solve for y in the first equation:

$$y=\frac{2-x\left(1+\sqrt{5}\right)}{1-\sqrt{5}}$$

and substitute into the second equation and solve for x:

$$\begin{aligned}
x\left(3+\sqrt{5}\right)+\frac{2-x\left(1+\sqrt{5}\right)}{1-\sqrt{5}}\left(3-\sqrt{5}\right) &= 2 && \text{multiply by } 1-\sqrt{5} \\
x\left(3+\sqrt{5}\right)\left(1-\sqrt{5}\right)+\left(2-x\left(1+\sqrt{5}\right)\right)\left(3-\sqrt{5}\right) &= 2\left(1-\sqrt{5}\right) \\
x\left(-2-2\sqrt{5}\right)+2\left(3-\sqrt{5}\right)-x\left(1+\sqrt{5}\right)\left(3-\sqrt{5}\right) &= 2-2\sqrt{5} \\
x\left(-2-2\sqrt{5}\right)+2\left(3-\sqrt{5}\right)-x\left(-2+2\sqrt{5}\right) &= 2-2\sqrt{5} \\
x\left(-2-2\sqrt{5}\right)+6-2\sqrt{5}+x\left(2-2\sqrt{5}\right) &= 2-2\sqrt{5} && \text{add } \sqrt{5} \text{ subtract } 6 \\
x\left(-2-2\sqrt{5}\right)+x\left(2-2\sqrt{5}\right) &= -4 && \text{factor out } x \\
x\left(-2-2\sqrt{5}+2-2\sqrt{5}\right) &= -4 \\
x\left(-4\sqrt{5}\right) &= -4 && \text{divide by } -4\sqrt{5} \\
x &= \frac{1}{\sqrt{5}}
\end{aligned}$$

We substitute this into the expression expressing y.

$$y=\frac{2-x\left(1+\sqrt{5}\right)}{1-\sqrt{5}} \quad \text{and} \quad x=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

$$\begin{aligned}
y &= \frac{2-\frac{1}{\sqrt{5}}\left(1+\sqrt{5}\right)}{1-\sqrt{5}} = \frac{2-\frac{1}{\sqrt{5}}-1}{1-\sqrt{5}} = \frac{1-\frac{1}{\sqrt{5}}}{1-\sqrt{5}} = \frac{1-\frac{\sqrt{5}}{5}}{1-\sqrt{5}}\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{\left(1-\frac{\sqrt{5}}{5}\right)\left(1+\sqrt{5}\right)}{1-5} \\
&= \frac{\left(\frac{5}{5}-\frac{\sqrt{5}}{5}\right)\left(1+\sqrt{5}\right)}{1-5} = \frac{\frac{1}{5}\left(5-\sqrt{5}\right)\left(1+\sqrt{5}\right)}{-4} = \frac{\left(5-\sqrt{5}\right)\left(1+\sqrt{5}\right)}{-20} = \frac{\sqrt{5}\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)}{-20} \\
&= \frac{\sqrt{5}\cdot 4}{-20} = -\frac{\sqrt{5}}{5}
\end{aligned}$$

So $x=\frac{1}{\sqrt{5}}$ and $y=-\frac{1}{\sqrt{5}}$. Consequently, the nth term in the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

This formula seems very unlikely to produce integers. Let us see the first few elements generated by the formula.

$$F_1 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \frac{1}{\sqrt{5}}\cdot\frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \frac{1}{\sqrt{5}}\cdot\frac{2\sqrt{5}}{2} = 1$$

$$\begin{aligned}
F_2 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right) = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}-\frac{3-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\cdot\frac{3+\sqrt{5}-3+\sqrt{5}}{2} \\
&= \frac{1}{\sqrt{5}}\cdot\frac{2\sqrt{5}}{2} = 1
\end{aligned}$$

Before computing F_3, let us compute $\left(\frac{1+\sqrt{5}}{2}\right)^3$.

$$\left(\frac{1+\sqrt{5}}{2}\right)^3 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right) = \frac{\left(1+\sqrt{5}\right)\left(3+\sqrt{5}\right)}{4} = \frac{8+4\sqrt{5}}{4} = 2+\sqrt{5}$$

We similarly obtain the exact value of $\left(\frac{1-\sqrt{5}}{2}\right)^3$ and then we are ready to compute F_3.

$$F_3 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right) = \frac{1}{\sqrt{5}}\left(\left(2+\sqrt{5}\right)-\left(2-\sqrt{5}\right)\right) = \frac{1}{\sqrt{5}}\cdot 2\sqrt{5} = 2$$