

Proof by Induction

Prove each of the following statements by induction.

1. For all natural numbers n ,

a) $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

b) $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

c) $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$.

2. For all natural numbers n , $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

3. For all natural numbers n , $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n = 2(1 + (n-1)2^n)$.

4. For all natural numbers $n \geq 5$, $2^n \geq n^2$.

5. Prove that for all natural number n , 10^n can be written as a sum of two perfect squares.

6. Recall (F_n) is the Fibonacci sequence, defined as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_n + F_{n+1}$$

Prove each of the following statements for all natural numbers n .

a) $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$

b) $F_1^2 + F_2^2 + F_3^2 + \dots + F_n^2 = F_n F_{n+1}$

c) $F_1 + F_3 + \dots + F_{2n-1} = F_{2n}$

d) $F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1$

e) For all $n \geq 2$, $F_n^2 - F_{n-1} \cdot F_{n+1} = (-1)^{n-1}$

7. Prove that for all natural number n , $11^n - 6$ is divisible by 5.

8. Prove that for all natural number n , $2^{3n} - 3^n$ is divisible by 5.

Sample Problems - Solutions

1. a) For all natural numbers n , $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

Proof:

Part 1: We check for the first few values of n .

If $n = 1$, then

$$\text{LHS} = 1 = 1 \quad \text{and} \quad \text{RHS} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

If $n = 2$, then

$$\text{LHS} = 1 + 2 = 3 \quad \text{and} \quad \text{RHS} = \frac{2(2+1)}{2} = \frac{6}{2} = 3$$

If $n = 3$, then

$$\text{LHS} = 1 + 2 + 3 = 6 \quad \text{and} \quad \text{RHS} = \frac{3(3+1)}{2} = \frac{12}{2} = 6$$

So the statement is true for $n = 1, 2$ and 3 .

Part 2. Suppose that k is a positive integer for which

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \text{This is the inductual hypotheses}$$

Let us add $k + 1$ to both sides. The left-hand side becomes

$$1 + 2 + 3 + \dots + k + (k + 1)$$

and the right hand side becomes

$$\begin{aligned} \text{RHS} &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

So we have proved that for all positive integers k , if

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \text{is true}$$

then

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$$

which completes our proof.

Proof by Induction

b) For all natural numbers n , $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof:

Part 1: We check for the first few values of n .

If $n = 1$, then

$$\text{LHS} = 1^2 = 1 \quad \text{and} \quad \text{RHS} = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1$$

If $n = 2$, then

$$\text{LHS} = 1^2 + 2^2 = 5 \quad \text{and} \quad \text{RHS} = \frac{2(2+1)(2 \cdot 2 + 1)}{6} = \frac{30}{6} = 5$$

If $n = 3$, then

$$\text{LHS} = 1^2 + 2^2 + 3^2 = 14 \quad \text{and} \quad \text{RHS} = \frac{3(3+1)(2 \cdot 3 + 1)}{6} = \frac{84}{6} = 14$$

So the statement is true for $n = 1, 2$ and 3 .

Part 2. Suppose that k is a positive integer for which

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{This is the inductual hypotheses}$$

Let us add $(k+1)^2$ to both sides. The left-hand side becomes

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$$

and the right hand side becomes

$$\text{RHS} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

We factor out $\frac{k+1}{6}$

$$\begin{aligned} \text{RHS} &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k+1}{6} (k(2k+1) + 6(k+1)) \\ &= \frac{k+1}{6} (2k^2 + k + 6k + 6) = \frac{k+1}{6} (2k^2 + 7k + 6) \end{aligned}$$

and so happens $2k^2 + 7k + 6$ factors as $(k+2)(2k+3)$. So we have

$$\begin{aligned} \text{RHS} &= \frac{k+1}{6} (2k^2 + 7k + 6) = \frac{k+1}{6} (k+2)(2k+3) = \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \end{aligned}$$

So we have proved that for all positive integers k , if

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

then

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

which completes our proof.

Proof by Induction

c) For all natural numbers n , $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Proof:

Part 1: We check for the first few values of n .

If $n = 1$, then

$$\text{LHS} = 1^3 = 1 \quad \text{and} \quad \text{RHS} = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

If $n = 2$, then

$$\text{LHS} = 1^3 + 2^3 = 9 \quad \text{and} \quad \text{RHS} = \frac{2^2(2+1)^2}{4} = \frac{36}{4} = 9$$

If $n = 3$, then

$$\text{LHS} = 1^3 + 2^3 + 3^3 = 36 \quad \text{and} \quad \text{RHS} = \frac{3^2(3+1)^2}{2^2} = \frac{144}{4} = 36$$

So the statement is true for $n = 1, 2$ and 3 .

Part 2. Suppose that k is a positive integer for which

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{This is the inductive hypothesis}$$

Let us add $(k+1)^3$ to both sides. The left-hand side becomes

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$$

and the right hand side becomes

$$\begin{aligned} \text{RHS} &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad \text{factor out } \frac{(k+1)^2}{4} \\ &= \frac{(k+1)^2}{4} (k^2 + 4(k+1)) = \frac{(k+1)^2}{4} (k^2 + 4k + 4) = \frac{(k+1)^2}{4} (k+2)^2 = \frac{(k+1)^2(k+1+1)^2}{4} \end{aligned}$$

So we have proved that for all positive integers k , if

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{is true}$$

then

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

which completes our proof.

Proof by Induction

2. For all natural numbers n , $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Proof: Induction on n .

Part 1. If $n = 1$, then

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2} \quad \text{and} \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

If $n = 2$, then

$$\text{LHS} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \quad \text{and} \quad \text{RHS} = \frac{2}{2+1} = \frac{2}{3}$$

Part 2. Suppose that k is a natural number such that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad (\text{Induction hypothesis})$$

We will add $\frac{1}{(k+1)(k+1+1)} = \frac{1}{(k+1)(k+2)}$ to both sides.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

Let us simplify the right hand side

$$\begin{aligned} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \end{aligned}$$

Thus we have that from the induction hypotheses

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

the statement

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

follows. This completes our proof.

3. For all natural numbers n , $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n = 2(1 + (n-1)2^n)$.

Proof:

Part 1. If $n = 1$, then

$$\text{LHS} = 1 \cdot 2 = 2 \quad \text{and} \quad \text{RHS} = 2(1 + (1-1)2^1) = 2 \cdot 1 = 2$$

If $n = 2$, then

$$\text{LHS} = 1 \cdot 2 + 2 \cdot 2^2 = 10 \quad \text{and} \quad \text{RHS} = 2(1 + (2-1)2^2) = 2 \cdot 5 = 10$$

Part 2. Suppose that k is a natural number such that

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + k \cdot 2^k = 2(1 + (k-1)2^k) \quad (\text{Induction hypothesis})$$

Let us add $(k+1)2^{k+1}$ to both sides. The left-hand side becomes

$$\text{LHS} = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + k \cdot 2^k + (k+1)2^{k+1}$$

Proof by Induction

and the right-hand side becomes

$$\begin{aligned}\text{RHS} &= 2\left(1 + (k-1)2^k\right) + (k+1)2^{k+1} = 2\left(1 + (k-1)2^k\right) + (k+1)2^k \cdot 2 && \text{factor out } 2 \\ &= 2\left(1 + (k-1)2^k + (k+1)2^k\right) = 2\left(1 + k2^k - 2^k + k2^k + 2^k\right) = \\ &= 2\left(1 + 2k2^k\right) = 2\left(1 + k2^{k+1}\right)\end{aligned}$$

which completes our proof.

4. For all natural numbers $n \geq 5$, $2^n \geq n^2$

Note: this is a very interesting example illustrating how induction works, how we need both parts to work together to form a logically sound proof. The statement seems to be true immediately at $n = 1$ but then surprisingly, it will be false for a few values of n .

Proof:

Part 1. If $n = 1$, then

$$\text{LHS} = 2^1 = 2 \quad \text{and} \quad \text{RHS} = 1^2 = 1 \quad \text{and} \quad 2 > 1 \quad \text{is true}$$

If $n = 2$, then

$$\text{LHS} = 2^2 = 4 \quad \text{and} \quad \text{RHS} = 2^2 = 4 \quad \text{and} \quad 4 > 4 \quad \text{is false!}$$

If $n = 3$, then

$$\text{LHS} = 2^3 = 8 \quad \text{and} \quad \text{RHS} = 3^2 = 9 \quad \text{and} \quad 8 > 9 \quad \text{is false!}$$

If $n = 4$, then

$$\text{LHS} = 2^4 = 16 \quad \text{and} \quad \text{RHS} = 4^2 = 16 \quad \text{and} \quad 16 > 16 \quad \text{is false!}$$

If $n = 5$, then

$$\text{LHS} = 2^5 = 32 \quad \text{and} \quad \text{RHS} = 5^2 = 25 \quad \text{and} \quad 32 > 25 \quad \text{is true}$$

If $n = 6$, then

$$\text{LHS} = 2^6 = 64 \quad \text{and} \quad \text{RHS} = 6^2 = 36 \quad \text{and} \quad 64 > 36 \quad \text{is true}$$

At this point we have a sense that the statement will stay true because the left-hand side doubles when we go from k to $k+1$ while the right-hand side just increases from k^2 to $(k+1)^2$. The proof in part 2 will formalize this idea, that between the two types of growth, doubling is much 'faster'.

All we need to show is that moving from k^2 to $(k+1)^2$ is a smaller increment than doubling. In short, that

$$\begin{aligned}(k+1)^2 &< 2k^2 \\ k^2 + 2k + 1 &< 2k^2 && \text{subtract } k^2 \\ 2k + 1 &< k^2\end{aligned}$$

This should be easy to prove for most positive integers. We can either solve the quadratic inequality or be a little bit sloppy or generous and say that if k is greater than 1, then

$$\begin{aligned}1 &< k && \text{add } 2k \\ 2k + 1 &< 2k + k \\ 2k + 1 &< 3k\end{aligned}$$

and if k is greater than 3, then

$$\begin{aligned}3 &< k && \text{multiply by } k > 0 \\ 3k &< k^2\end{aligned}$$

Proof by Induction

In short, if $k > 3$, then $2k + 1 < 3k < k^2$ and so

$$\begin{aligned} 2k + 1 &< k^2 && \text{add } k^2 \text{ to both sides} \\ k^2 + 2k + 1 &< 2k^2 \\ (k + 1)^2 &< 2k^2 \end{aligned}$$

This will be the core of the proof in part 2.

Part 2. Suppose that k is a natural number such that

$$2^k > k^2 \quad (\text{Induction hypothesis})$$

Let us multiply both sides by 2. The left-hand side becomes

$$\text{LHS} = 2 \cdot 2^k = 2^{k+1}$$

and the right-hand side becomes

$$\text{RHS} = 2k^2$$

So we have

$$\begin{aligned} 2^k &> k^2 \\ 2^{k+1} &> 2k^2 \end{aligned}$$

If $k > 3$, then

$$2^{k+1} > 2k^2 = k^2 + k^2 > k^2 + 3k > k^2 + 2k + 1 = (k + 1)^2$$

and so we have that if $2^k > k^2$ is true AND $k > 3$, then $2^{k+1} > (k + 1)^2$.

So we could easily prove the 'inheritance' property for all integers greater than 3 but the statement itself is NOT true for $n = 3$. Both components of the proof work together starting at $n = 5$.

	statement	inheritance to the next number
$n = 1$	true	false
$n = 2$	false	false
$n = 3$	false	true (but we didn't prove it)
$n = 4$	false	true
$n = 5$	true	true

This table illustrates why induction only works here for $n \geq 5$.