Let a and b represent positive numbers. The arithmetic, geometric, and harmonic means of a and b are defined as follows.

$$\begin{aligned} \text{arithmetic mean} &= \frac{a+b}{2} \\ \text{geometric mean} &= \sqrt{ab} \\ \text{harmonic mean} &= \frac{2ab}{a+b} \quad \text{or} \quad \frac{2}{\frac{1}{a}+\frac{1}{b}} \end{aligned}$$

As it turns out, all three of these means occur in mathematics and physics.

For any a and b, these three means have a natural order. The arithmetic mean is always the largest, and the harmonic mean is always the smallest. In short,

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$

and the equality holds if and only if $a=b$.

Theorem 1 *(The arithmetic and geometric means). Suppose that a and b are positive numbers. Then $\frac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a=b$.*

Proof: For all a and b, $(a-b)^2 \geq 0$ and the equality holds if and only if $a=b$.

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2-2ab+b^2 &\geq 0 \qquad \text{add } 4ab \\ a^2+2ab+b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab \end{aligned}$$

at this point, we take the square root of both sides. It is important to note that what allow this step, is that both a and b are positive.

$$\begin{aligned} \sqrt{(a+b)^2} &\geq \sqrt{4ab} \\ a+b &\geq 2\sqrt{ab} \qquad \text{divide by 2} \\ \frac{a+b}{2} &\geq \sqrt{ab} \end{aligned}$$

Theorem 2 *(The geometric and harmonic means.) Suppose that a and b are positive numbers. Then $\frac{2ab}{a+b} \leq \sqrt{ab}$ and the equality holds if and only if $a=b$.*

Proof: This statement is true because the previous one is true. Starting with that statement,

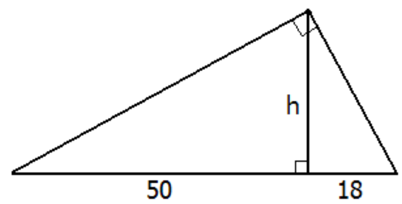
$$\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab} \qquad \text{multiply by 2} \\ a+b &\geq 2\sqrt{ab} \qquad \text{divide by } a+b \\ 1 &\geq \frac{2\sqrt{ab}}{a+b} \qquad \text{multiply by } \sqrt{ab} \\ \sqrt{ab} &\geq \frac{2ab}{a+b} \end{aligned}$$

Exercises

1. Find all three means for $a = 36$ and $b = 64$.

2. Prove that the two forms of the harmonic mean are equivalent.

3. The picture below shows a right triangle. Find the length of the height drawn to the hypotenuse.



4. A bus travels between cities A and B. From A to B, the bus has an average speed of v_1. On its way back, the average speed is v_2. Express the average speed of the bus in terms of v_1 and v_2.

5. Prove that for any positive number, the sum of the number and its reciprocal is at least 2. For what numbers is this sum exactly 2?

Answers to Exercises

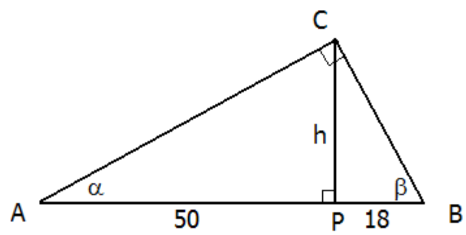
1. Arithmetic Mean: $\dfrac{36+64}{2}=50$

   Geometric Mean: $\sqrt{36\cdot 64}=48$

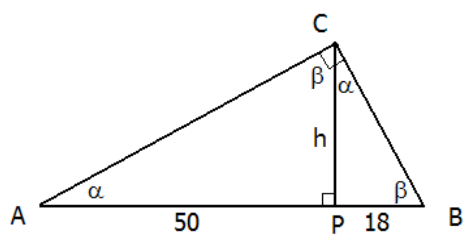
   Harmonic Mean: $\dfrac{2\,(36)\,64}{36+64}=46.08$

2. $\dfrac{2}{\dfrac{1}{a}+\dfrac{1}{b}}=\dfrac{2}{\dfrac{b+a}{ab}}=2\cdot\dfrac{ab}{a+b}=\dfrac{2ab}{a+b}$

3. Solution: Let us first label the points, angles and sides in the triangle.



Since ABC triangle is a right triangle, we have that $\alpha+\beta=90°$. Because of this, angle ACP must be equal to β, and angle PCB is equal to α. Thus the height drawn to the hypotenuse splits the original triangle into two triangles that have identical angles as the original triangle. Thus, all three triangles, $\triangle ABC$, $\triangle APC$ and $\triangle PBC$ are similar.



Consider now the ratio $\dfrac{\text{side opposite angle }\beta}{\text{side opposite angle }\alpha}$ in triangles $\triangle APC$ and $\triangle PBC$. Since these triangles are similar, this ratio is preserved.

$$\frac{\text{side opposite angle }\beta}{\text{side opposite angle }\alpha}=\frac{50}{h}=\frac{h}{18}$$

We solve this equation for h :

$$\begin{aligned}\frac{50}{h}&=\frac{h}{18}\\ 50\cdot 18&=h^2\\ 900&=h^2\\ h&=\pm 30\end{aligned}$$

$h=-30$ is ruled out since distances can not be negative. Thus $h=\sqrt{18\cdot 50}=30$, the geometric mean of 18 and 50.

4. Let t_1 and v_1 denote the time and speed associated with the trip from A to B, and t_2 and v_2 the time and speed associated with the trip from B to A. In both cases, the distance will be denoted by s.

$$v_{\text{av}} = \frac{\text{distance traveled}}{\text{time}} = \frac{s+s}{t_1+t_2} = \frac{2s}{\dfrac{s}{v_1}+\dfrac{s}{v_2}} = \frac{2s}{\dfrac{sv_2+sv_1}{v_1v_2}} = 2s\cdot\frac{v_1v_2}{s\left(v_1+v_2\right)} = \frac{2v_1v_2}{v_1+v_2}$$

The average speed on the roundtrip is $\dfrac{2v_1v_2}{v_1+v_2}$, the harmonic average of the individual speeds.

5. Solution: Let x be a positive number. We state the arithmetic-geometric mean theorem for x and $\dfrac{1}{x}$.

$$\begin{aligned} \frac{x+\frac{1}{x}}{2} &\geq \sqrt{x\cdot\frac{1}{x}} \\ \frac{x+\frac{1}{x}}{2} &\geq 1 && \text{multiply by 2} \\ x+\frac{1}{x} &\geq 2 \end{aligned}$$

The equality holds if x and $\dfrac{1}{x}$ are equal.

$$\begin{aligned} x &= \frac{1}{x} && \text{multiply by } x, \quad (x>0) \\ x^2 &= 1 \\ x &= \pm 1 \\ x &= 1 && \text{since } x>0 \end{aligned}$$

Thus only 1 is a number with the property that the sum of it and its reciprocal is exactly 2. For all other numbers, this sum is greater than 2.

