Definition: Let a and b represent positive numbers. The arithmetic and geometric, and harmonic means of a and b are defined as follows.

$$\text{arithmetic mean} = \frac{a+b}{2} \quad \text{and} \quad \text{geometric mean} = \sqrt{ab}$$

For any a and b, these means have a natural order. The arithmetic mean is always greater than or equal to the geometric mean: $\dfrac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a = b$.

Theorem: (The arithmetic and geometric means). Suppose that a and b are positive numbers. Then

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and the equality holds if and only if $a = b$.

Proof: Recall that a and b are both positive, so $\sqrt{a}$ and $\sqrt{b}$ both exist. For all a and b, $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$ and the equality holds if and only if $a = b$.

$$\begin{aligned}
\left(\sqrt{a}-\sqrt{b}\right)^2 &\geq 0 \\
\left(\sqrt{a}\right)^2 - 2\sqrt{a}\sqrt{b} + \left(\sqrt{b}\right)^2 &\geq 0 \\
a - 2\sqrt{ab} + b &\geq 0 && \text{add } 2\sqrt{ab} \\
a + b &\geq 2\sqrt{ab} && \text{divide by } 2 \\
\frac{a+b}{2} &\geq \sqrt{ab}
\end{aligned}$$

The inequality between arithmetic and geometric means has many applications. Here is one example.

Example 1. Prove that for all positive numbers x, $x + \dfrac{1}{x} \geq 2$.

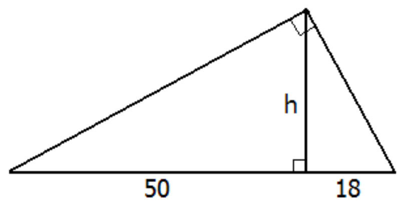
Solution: Let us apply the arithmetic-geometric means to the positive numbers x and $\dfrac{1}{x}$. The arithmetic mean of these two numbers is $\dfrac{x+\frac{1}{x}}{2}$, and their geometric mean is $\sqrt{x \cdot \dfrac{1}{x}} = \sqrt{1} = 1$. Now we state the inequality:

$$\begin{aligned}
\frac{x+\frac{1}{x}}{2} &\geq 1 && \text{multiply by } 2 \\
x + \frac{1}{x} &\geq 2
\end{aligned}$$

This completes our proof.

Exercises

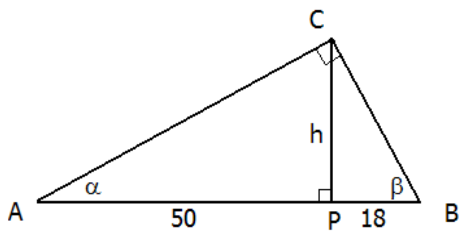
1. Find both means for $a = 36$ and $b = 64$.
2. The picture below shows a right triangle. Find the length of the height drawn to the hypotenuse.



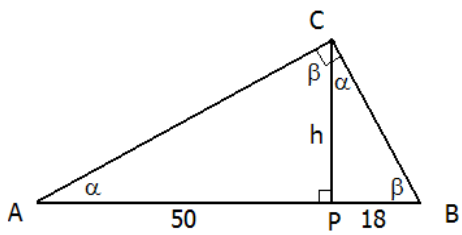
Answers to Exercises

1. Arithmetic Mean: $\dfrac{36+64}{2} = 50$ and geometric Mean: $\sqrt{36 \cdot 64} = 48$

2. Solution: Let us first label the points, angles and sides in the triangle.



Since ABC triangle is a right triangle, we have that $\alpha + \beta = 90°$. Because of this, angle ACP must be equal to β, and angle PCB is equal to α. Thus the height drawn to the hypotenuse splits the original triangle into two triangles that have identical angles as the original triangle. Thus, all three triangles, $\triangle ABC$, $\triangle APC$ and $\triangle PBC$ are similar.



Consider now the ratio $\dfrac{\text{side opposite angle } \beta}{\text{side opposite angle } \alpha}$ in triangles $\triangle APC$ and $\triangle PBC$. Since these triangles are similar, this ratio is preserved.

$$\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \alpha} = \frac{50}{h} = \frac{h}{18}$$

We solve this equation for h :

$$\begin{aligned} \frac{50}{h} &= \frac{h}{18} \\ 50 \cdot 18 &= h^2 \\ 900 &= h^2 \\ h &= \pm 30 \end{aligned}$$

$h = -30$ is ruled out since distances can not be negative. Thus $h = \sqrt{18 \cdot 50} = 30$, the geometric mean of 18 and 50.

