Sample Problems

1. We are standing on the top of a 720 ft tall building and throw a small object upward. The object's distance from the ground, measured in feet, after t seconds is
$$h(t) = -16t^2 + 192t + 720$$
What is the highest point that the object reaches?

2. A company finds that if they price their product at \$60, they can sell 500 items of it. For every dollar increase in the price, the number of items sold will decrease by 5.

   a) Let x be the increase in price from \$60. Define the revenue function, $R(x)$ to be the sales revenue that results in such pricing. Find a formula for $R(x)$.
   b) What price would guarantee an income of \$31 500?
   c) Find the price that guarantees the maximum revenue.
   d) Find the maximum revenue.

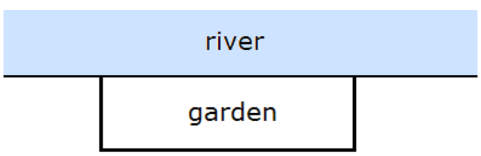
3. The cost of manufacturing q units of a product is given by $C(q) = 6q^2 + 10q$. Suppose we can sell all q units for a price of $p(q) = 142 + \dfrac{1674}{q}$ dollars. Find the maximum profit we can achieve.

4. Among the rectangles of perimeter 12 m, which one has the largest area?

5. A manufacturer estimates that when q units of a particular commodity are produced each month, the total cost will be $C(q) = 0.4q^2 + 3q + 40$ thousand dollars, and all units can be sold at a price of $p(q) = 22.2 - 1.2q$ thousand dollars per unit. Find the maximal profit.

6. It costs 12 dollars each to manufacture and distribute a certain product. If we price it at x dollars each, the number sold is approximated by $n(x) = \dfrac{60}{x-12} + \dfrac{1}{5}(100 - x)$. What selling price will bring in the maximum profit? How much money is that?

7. A gardener has 120 feet of fencing to fence in a rectangular garden. One side of the garden is bordered by a river and so it does not need any fencing.



   a) What dimensions would guarantee a garden with an area of $1350\,\text{ft}^2$?
   b) What dimensions would guarantee the greatest area? How much is the greatest area?

8. Let a and b be any real numbers such that $a + b = 10$. Prove that $a^2 + b^2 \geq 50$.

9. Let m and n be two numbers with $m + 3n = 20$. Find

   a) the smallest value of $m^2 + n^2$
   b) the greatest value of $n^2 - m^2$
   c) the greatest value of mn

10. Let a and b be any number such that $a + b = S$. Find the smallest possible value of $a^2 + b^2$.

11. Suppose that x, y, r and s are real numbers such that $x + y = 12$ and $r + s = 6$. Prove that then $(xr + ys)^2 + (xs - yr)^2 \geq 1296$.

Practice Problems

1. We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is
$$h(t) = 160t - 16t^2 + 1200$$
a) Where is the object 3 seconds after we threw it?
b) How long does it take for the object to hit the ground?
c) What is the highest point that the object reaches?

2. If we set the price of our product to be \$18 per item, then we can sell 300 items. For every dollar we raise the price, we can sell 5 less items.
a) Let x be the increase in price from \$18. Define the revenue function, $R(x)$ to be the sales revenue that results in such pricing. Find a formula for $R(x)$.
b) How much is the total income if we price the product at \$24?
c) What price woud guarantee an income of \$6625?
d) What price would guarantee the possible highest income? What is the highest possible income?

3. A citrus grower estimates that if 60 orange trees are planted, the average yield per tree will be 400 oranges. The average yield will decrease by 4 oranges per tree for each additional tree planted on the same acreage. Find the total number of trees the grower should plant to maximize yield.

4. A farmer wishes to enclose a rectangular pasture with 320 ft of fence. What dimensions give the maximum area if the fence is on three sides of the pasture and the fourth side is bounded by a wall?

5. Side AB of a rectangle $ABCD$ is located on the $x-$ axis, within the interval $[-6, 6]$. Another horizontal side is defined by points C and D, lying on the straight lines $y = 2x + 12$ and $y = -2x + 12$, respectively. Find the maximal possible area of the rectangle.

6. A manufacturer estimates that when q units of a particular product are produced each month, the total cost will be $C(q) = 5q + 17000$ dollars, and all q units can be sold at a price of $p(q) = 65 - \dfrac{q}{100}$ dollars per unit.
a) Find the fixed cost.
b) Determine the level of production that results in a maximum profit.
c) What is the maximum profit?

Sample Problems - Answers

1. 1296 ft
2. a) $R(x) = (60 + x)(500 - 5x)$ b) \$70 or \$90 c) \$80 d) \$32 000
3. \$2400
4. the square
5. \$17 600
6. \$440 with pricing at \$56
7. a) 15 ft by 90 ft or 45 ft by 30 ft b) sides: 30 ft and 60 ft; area: $A = 1800\,\text{ft}^2$
8. see solutions
9. a) 40 when $n = 6$ and $m = 2$ b) 50 when $n = \frac{15}{2}$ and $m = -\frac{5}{2}$ c) $\frac{100}{3}$ when $n = \frac{10}{3}$ and $m = 10$
10. $\frac{S^2}{2}$
11. see solutions

Practice Problems - Answers

1. a) 1536 ft b) 15 seconds c) 1600 ft, after 5 seconds
2. a) $R(x) = (18 + x)(300 - 5x)$ b) \$6480 c) \$25 and \$53 d) \$7605, with a price of \$39
3. 25 600 oranges, with 80 trees
4. 80 ft by 160 ft
5. 36 unit^2
6. a) \$17 000 b) 3000 unit per month c) \$73 000

Sample Problems - Solutions

1. We are standing on the top of a 720 ft tall building and throw a small object upward. The object's distance, measured in feet, after t seconds is
$$h(t) = -16t^2 + 192t + 720$$
What is the highest point that the object reaches?
Solution. $h(t)$ is quadratic, with a negative leading coefficient. Consequently, the graph of $h(t)$ is an upside down turned parabola. The highest point is the vertex. We will find the vertex by completing the square.
$$\begin{aligned} 0 &= -16t^2 + 192t + 720 = -16\left(t^2 - 12t - 45\right) && (t-6)^2 = t^2 - 12t + 36 \\ 0 &= -16\left(t^2 - 12t + 36 - 36 - 45\right) \\ 0 &= -16\left((t-6)^2 - 81\right) && \text{distribute } -16 \\ h(t) &= -16(t-6)^2 + 1296 \end{aligned}$$
This indicates that the upside down parabola $h(t)$ has its vertex at $(6, 1296)$. Thus the highest point is reached 6 seconds after we threw the object, and it is 1296 feet high.

2. A company finds that if they price their product at \$60, they can sell 500 items of it. For every dollar increase in the price, the number of items sold will decrease by 5.

a) Let x be the increase in price from \$60. Define the revenue function, $R(x)$ to be the sales revenue that results in such pricing. Find a formula for $R(x)$.
Solution: Revenue = (Price) · (Number of items sold)
$$\begin{aligned} R(x) &= (60+x)(500-5x) && \text{rearrange} \\ &= (x+60)(-5x+500) && \text{factor out } -5 \\ &= -5(x+60)(x-100) \\ &= -5\left(x^2 - 100x + 60x - 6000\right) \\ &= -5\left(x^2 - 40x - 6000\right) \end{aligned}$$

b) What price would guarantee an income of \$31 500?
Solution: We have to solve the equation: $x = ?$ so that $R(x) = 31500$.
$$x = ? \quad \text{so that} \quad R(x) = 31500$$
$$\text{Solve} \quad -5x^2 + 200x + 30000 = 31500 \quad \text{for } x$$
Since the equation is quadratic, we will reduce one side to zero. We will avoid a negative leading coefficient by reducing the left-hand side to zero.
$$\begin{aligned} -5x^2 + 200x + 30000 &= 31500 \\ 0 &= 5x^2 - 200x + 1500 \end{aligned}$$

We will factor by completing the square.

$$\begin{aligned}
5x^2 - 200x + 1500 &= 0 \quad \text{factor out } 5 \\
5\left(x^2 - 40x + 300\right) &= 0 \qquad\qquad (x-20)^2 = x^2 - 40x + 400 \\
5\left(\underbrace{x^2 - 40x + 400} - 400 + 300\right) &= 0 \\
5\left((x-20)^2 - 100\right) &= 0 \\
5\left((x-20)^2 - 10^2\right) &= 0 \\
5(x - 20 + 10)(x - 20 - 10) &= 0 \\
5(x-10)(x-30) &= 0 \quad \Longrightarrow \quad x_1 = 10 \qquad x_2 = 30
\end{aligned}$$

We check. If $x = 10$, the price is increased by \$10 and so it sells for \$70. We can then sell $500 - 5(10) = 450$ items, and so the income is $R(10) = 70 \cdot 450 = 31\,500$. On the other hand, if $x = 30$, the selling price is \$ 90. We then can sell $500 - 5(30) = 350$ items, and so the revenue is $R(30) = 90 \cdot 350 = 31\,500$.

c) Find the price that guarantees the maximum revenue.
Solution: $R(x)$ is clearly a downward turning parabola. The maximum $y-$value is the y-value of the vertex. To find the vertex, we complete the square.

$$\begin{aligned}
R(x) &= -5x^2 + 200x + 30000 \\
&= -5\left(x^2 - 40x - 6000\right) \qquad\qquad (x-20)^2 = x^2 - 40x + 400 \\
&= -5\left(\underbrace{x^2 - 40x + 400} - 400 - 6000\right) \\
&= -5\left((x-20)^2 - 6400\right) \qquad\qquad \text{distribute } -5 \\
&= -5(x-20)^2 + 32\,000
\end{aligned}$$

The vertex of the parabola is $(20, 32\,000)$. Thus a \$20 increase, implying a price of \$80, will guarantee a maximal income of \$32 000.

d) Find the maximum revenue. Solution: see part c).

3. The cost of manufacturing q units of a product is given by $C(q) = 6q^2 + 10q$. Suppose we can sell all q units for a price of $p(q) = 142 + \dfrac{1674}{q}$ dollars. Find the maximum profit we can achieve.
Solution: Revenue = (Price) $\cdot$ (Number of items sold) and Profit = Revenue - Cost

$$\begin{aligned}
\Pr(q) &= R(q) - C(q) = q \cdot p(q) - C(q) \\
&= q\left(142 + \frac{1674}{q}\right) - \left(6q^2 + 10q\right) = (142q + 1674) - \left(6q^2 + 10q\right) \\
&= -6q^2 + 132q + 1674 \\
&= -6\left(q^2 - 22q - 279\right) \qquad\qquad (q-11)^2 = q^2 - 22q + 121 \\
&= -6\left(\underbrace{q^2 - 22q + 121} - 121 - 279\right) \\
&= -6\left((q-11)^2 - 400\right) = -6(q-11)^2 + 2400
\end{aligned}$$

As the computation shows, this is an upside down parabola, so its maximum is its vertex, $(11, 2400)$. Thus the maximum profit is \$2400.

4. Among the rectangles of perimeter 12 m, which one has the largest area?
Solution: Let us denote the sides of the rectangle by x and y. Then

$$\begin{aligned} 12 &= 2(x+y) \quad \text{solve for } y \\ 12 &= 2x+2y \\ 12-2x &= 2y \\ y &= \frac{12-2x}{2} = 6-x \end{aligned}$$

Then the area of the rectangle, as a function of x is

$$A(x) = xy = x(6-x) = -x^2+6x$$

This is an upside-down parabola that has a maximum at its vertex. We find the vertex by completing the square.

$$\begin{aligned} A(x) &= -x^2+6x = -1\left(x^2-6x\right) = -1\left(x^2-6x+9-9\right) \\ &= -1\left((x-3)^2-9\right) = -1(x-3)^2+9 \end{aligned}$$

The last line tells us that the vertex of $A(x)$ is $(3,9)$ which means that the largest area is $9\,\text{m}^2$, when the sides are $x=3\,\text{m}$ and $y=6-3=3\,\text{m}$.

5. A manufacturer estimates that when q units of a particular commodity are produced each month, the total cost will be $C(q) = 0.4q^2+3q+40$ thousand dollars, and all units can be sold at a price of $p(q) = 22.2-1.2q$ thousand dollars per unit. Find the maximal profit.
Solution: Profit = Revenue − Cost = (Price) · (Number Sold) −Cost. As a function of q, the profit is

$$\begin{aligned} \text{Profit}(q) &= (22.2-1.2q)(1000)(q) - \left(0.4q^2+3q+40\right)(1000) \\ &= q(22\,200-1\,200q) - \left(400q^2+3\,000q+40\,000\right) \\ &= 22\,200q-1\,200q^2-400q^2-3\,000q-40\,000 \\ &= -1\,600q^2+19\,200q-40\,000 \end{aligned}$$

This polynomial is quadratic, with a negative leading coefficient. Thus its graph is a downward turning parabola. Such a graph has a maximum at its vertex. To find the vertex, we complete the square.

$$\begin{aligned} \text{Profit}(q) &= -1\,600q^2+19\,200q-40\,000 \\ &= -1600\left(q^2-12q+25\right) \\ &= -1600\left(\underbrace{q^2-12q+36}-36+25\right) \\ &= -1600\left((q-6)^2-11\right) \\ &= -1600(q-6)^2+17\,600 \end{aligned}$$

The vertex is $(6, 17\,600)$. This indicates a maximum profit of \$17 600, when 6 items sold.

6. It costs 12 dollars each to manufacture and distribute a certain product. If we price it at x dollars each, the number sold is approximated by $n(x) = \dfrac{60}{x-12} + \dfrac{1}{5}(100-x)$. What selling price will bring in the maximum profit? How much money is that?
Solution: Let $P(x)$ denote the profit as a functon of f. We need to find the maximum for P.

$$\begin{aligned}
\text{Profit} &= \text{Revenue} - \text{Cost} = (\text{Price})(\text{Number Sold}) - \text{Cost} \\
P(x) &= n(x)\cdot x - n(x)\cdot 12 = n(x)\cdot(x-12) \\
&= \left(\frac{60}{x-12} + \frac{1}{5}(100-x)\right)(x-12) \\
&= 60 + \frac{1}{5}(100-x)(x-12) = 60 + \frac{1}{5}\left(-x^2+112x-1200\right) \\
&= 60 - \frac{1}{5}x^2 - \frac{112}{5}x - 240 = -\frac{1}{5}x^2 + \frac{112}{5}x - 180 \\
&= -\frac{1}{5}\left(x^2 - 112x + 900\right) \\
&= -\frac{1}{5}\left(\underbrace{x^2 - 112x + 3136}\, -3136 + 900\right) \qquad 56^2 = 3136 \\
&= -\frac{1}{5}\left((x-56)^2 - 2236\right) \\
&= -\frac{1}{5}(x-56)^2 + \frac{2236}{5} \qquad \Longrightarrow \text{Vertex } \left(56, \frac{2236}{5}\right)
\end{aligned}$$

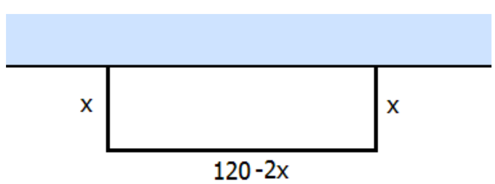
Thus the maximal profit occurs when the price is \$56. However, it is not $P(56) = \dfrac{2236}{5} = 447.20$. This would be a wrong answer because it computes the profit in terms of a fractional number of items sold, since

$$n(56) = \frac{60}{56-12} + \frac{1}{5}(100-56) = \frac{559}{55} = 10.1\overline{63}$$

The best answer can be obtained by using a price of \$56 and 10 as the number of items sold. Thus the maximum revenue is

$$P = (\text{Price})(\text{Number Sold}) - \text{Cost} = \$56\cdot 10 - \$12\cdot 10 = \$440$$

7. A gardener has 120 feet of fencing to fence in a rectangular garden. One side of the garden is bordered by a river and so it does not need any fencing.



a) What dimensions would guarantee a garden with an area of $1350\,\text{ft}^2$?
Solution: Let us denote the side perpendicular to the river by x. Then the other side is $120-2x$. The area of the rectangle is then $A(x) = x(120-2x)$. We simply solve the quadratic equation $A(x) = 1350$.

$$\begin{aligned} x(120-2x) &= 1350 \\ -2x^2+120x &= 1350 \\ 0 &= 2x^2-120x+1350 \\ 0 &= 2\left(x^2-60x+675\right) \\ 0 &= 2\left(\underbrace{x^2-60x+900}-900+675\right) \\ 0 &= 2\left((x-30)^2-225\right) \qquad\qquad 225=15^2 \\ 0 &= 2(x-30+15)(x-30-15) \\ 0 &= 2(x-15)(x-45) \quad\Longrightarrow\quad x_1=15 \qquad x_2=45 \end{aligned}$$

The first solution, $x=15$ means that the other side is $120-2(15)=90$ ft long. The second solution, $x=45$ means that the other side is $120\text{ ft}-2(45\text{ ft})=30$ ft long. Thus, there are two possible answers: a rectangle with dimensions 15 ft by 90 ft, or with dimensions 45 ft by 30 ft. They both are correct solutions.

b) What dimensions would guarantee the greatest area? How much is the greatest area?
Solution: Let us denote the side perpendicular to the river by x. Then the other side is $120-2x$. The area of the rectangle is then $A(x)=x(120-2x)$. We need to find the maximum of $A(x)$. This exists, since $A(x)$ is a quadratic function with a negative leading coefficient, thus its graph is a downward opening parabola and so its vertex is a maximum. We find the vertex by completing the square.

$$\begin{aligned} A(x) &= x(120-2x) = -2x^2+120x = -2\left(x^2-60x\right) \\ &= -2\left(\underbrace{x^2-60x+900}-900\right) = -2\left((x-30)^2-900\right) \\ &= -2(x-30)^2+1800 \end{aligned}$$

The last line tells us that the vertex of this parabola is $(30, 1800)$. Thus, the perpendicular side being 30 ft (and the other side $120\text{ ft}-2(30\text{ ft})=60\text{ ft}$) will result in a maximal area of 1800 ft^2.

8. Let a and b be any real numbers such that $a+b=10$. Prove that $a^2+b^2\geq 50$.
Solution: We express b in terms of a. Clearly, $b=10-a$. Then

$$\begin{aligned} a^2+b^2 &= a^2+(10-a)^2 = a^2+a^2-20a+100 = 2a^2-20a+100 \\ &= 2\left(a^2-10a+50\right) = 2\left(\underbrace{a^2-10a+25}-25+50\right) \\ &= 2\left((a-5)^2+25\right) \geq 2\cdot 25 = 50 \end{aligned}$$

9. Let m and n be two numbers with $m+3n=20$. Find

a) the smallest value of m^2+n^2
Solution: We solve for m: $m=20-3n$. We substitute this into the expression m^2+n^2.

$$m^2+n^2 = (20-3n)^2+n^2 = 9n^2-120n+400+n^2 = 10n^2-120n+400$$

Since this is a regular parabola, its minimum is at its vertex. We complete the square to find this minimum.

$$\begin{aligned} 10n^2-120n+400 &= 10\left(n^2-12n+40\right) = 10\left(\underbrace{n^2-12n+36}-36+40\right) = \\ &= 10\left((n-6)^2+4\right) = 10(n-6)^2+40 \end{aligned}$$

This means that the smallest value is 40 when $n=6$. Then $m=20-3n=20-3\cdot 6=2$.

b) the greatest value of n^2-m^2

Solution: We substitute $m=20-3n$ into the expression n^2-m^2.

$$\begin{aligned} n^2-m^2 &= n^2-(20-3n)^2=n^2-\left(9n^2-120n+400\right)=n^2-9n^2+120n-400 \\ &= -8n^2+120n-400=-8\left(n^2-15n+50\right)=-8\left(n^2-15n+\left(\frac{15}{2}\right)^2-\left(\frac{15}{2}\right)^2+50\right) \\ &= -8\left(\left(n-\frac{15}{2}\right)^2-\frac{225}{4}+\frac{200}{4}\right)=-8\left(\left(n-\frac{15}{2}\right)^2-\frac{25}{4}\right)=-8\left(n-\frac{15}{2}\right)^2+50 \end{aligned}$$

This means that the greatest value is 50 when $n=\frac{15}{2}$.

Then $m=20-3n=m=20-3\left(\frac{15}{2}\right)=\frac{40-45}{2}=-\frac{5}{2}$.

c) the greatest value of mn

Solution: We substitute $m=20-3n$ into the expression mn.

$$\begin{aligned} mn &= (20-3n)\,n=-3n^2+20n=-3\left(n^2-\frac{20}{3}n\right)=-3\left(n^2-\frac{20}{3}n+\frac{100}{9}-\frac{100}{9}\right) \\ &= -3\left(\left(n-\frac{10}{3}\right)^2-\frac{100}{9}\right)=-3\left(n-\frac{10}{3}\right)^2+\frac{100}{3} \end{aligned}$$

This means that the greatest value is $\frac{100}{3}$ when $n=\frac{10}{3}$.

Then $m=20-3n=m=20-3\left(\frac{10}{3}\right)=10$.

10. Let a and b be any number such that $a+b=S$. Find the smallest possible value of a^2+b^2.

Solution: $b=S-a$

$$\begin{aligned} a^2+b^2 &= a^2+(S-a)^2=a^2+a^2-2Sa+S^2=2a^2-2Sa+S^2=2\left(a^2-Sa+\frac{S^2}{2}\right) \\ &= 2\left(\underbrace{a^2-Sa+\frac{S^2}{4}}-\frac{S^2}{4}+\frac{S^2}{2}\right)=2\left(\underbrace{a^2-Sa+\frac{S^2}{4}}+\frac{S^2}{4}\right) \\ &= 2\left(\left(a-\frac{S}{2}\right)^2+\frac{S^2}{4}\right)=2\left(a-\frac{S}{2}\right)^2+\frac{S^2}{2}\geq\frac{S^2}{2} \end{aligned}$$

11. Suppose that x,y,r and s are real numbers such that $x+y=12$ and $r+s=6$. Prove that then $(xr+ys)^2+(xs-yr)^2\geq 1296$.

Solution: Let us multiply out the expression on the left-hand side.

$$\begin{aligned} (xr+ys)^2+(xs-yr)^2 &= (xr)^2+(ys)^2+2\,(xr)\,(ys)+(xs)^2+(yr)^2-2\,(xs)\,(yr) \\ &= x^2r^2+y^2s^2+2xyrs+x^2s^2+y^2r^2-2xyrs \\ &= x^2\left(r^2+s^2\right)+y^2\left(r^2+s^2\right)=\left(x^2+y^2\right)\left(r^2+s^2\right) \end{aligned}$$

We can prove (see the previous problem) that $x^2+y^2\geq 72$ and $r^2+s^2\geq 18$ and so we have that $\left(x^2+y^2\right)\left(r^2+s^2\right)\geq 72\cdot 18=1296$.

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