

Sample Problems

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x .
2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x .
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 9x + 7$ with slope -3 .
4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0, 1)$.
5. Find an equation for all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 - 6x + 30$ from the point $(7, 8)$.
6. Find an equation for all tangent lines drawn to the graph of $y = x^2 - 10x + 16$ from the point $(7, -14)$.

Practice Problems

1. Find the value of p so that the quadratic equation $x^2 + 10x + 3p - 8 = 0$ has exactly one solution for x .
2. Consider the parametric equation $x^2 - 6x + 49 = 2ax + 12a$. Find all values of a for which the equation has one real solution for x .
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 17x + 59$ with slope -3 .
4. In each case, find an equation for all tangent lines drawn to the given graph from the point given.
 - a) $y = x^2 - 8x + 3$ from $P(0, -6)$
 - b) $f(x) = 7x - \frac{1}{2}x^2 - 20$ from $P(4, 8)$
 - c) $y = \frac{1}{2}x^2 + 3x - 11$ from $P(2, -5)$
 - d) $f(x) = \frac{1}{2}x^2 - x + 2$ from $P(3, -1)$

Sample Problems - Answers

1. 3
2. 7, -3
3. $y = -3x - 2$
4. $y = x + 1$ and $y = 5x + 1$
5. $y = 4x - 20$ and $y = -2x + 22$
6. $y = -2x$ and $y = 10x - 84$

Practice Problems – Answers

1. 11
2. -20, 2
3. $y = -3x + 10$
4. In each case, find an equation for all tangent lines drawn to the given graph from the point given.
 - a) $y = -2x - 6$ and $y = -14x - 6$
 - b) $y = -x + 12$ and $y = 7x - 20$
 - c) $y = 3x - 11, y = 7x - 19$
 - d) $y = -x + 2$ and $y = 5x - 16$

Sample Problems – Solutions

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x .
 Solution: We complete the square

$$\begin{aligned} x^2 - 8x + 6m &= 2 \\ x^2 - 8x + 6m - 2 &= 0 \\ \underbrace{x^2 - 8x + 16} - 16 + 6m - 2 &= 0 \\ (x - 4)^2 + 6m - 18 &= 0 \end{aligned}$$

For all values of m , this expression is quadratic. If we look at this as a parabola, its vertex has

$$\begin{aligned} x - \text{coordinate:} & \quad 4 \\ y - \text{coordinate:} & \quad 6m - 18 \end{aligned}$$

for exactly one solution (or x -intercept), the vertex needs to be on the x -axis. In other words, this quadratic equation has exactly one solution if the quantity added to the complete square is zero. This gives us an equation we can solve for m .

$$\begin{aligned} 6m - 18 &= 0 \\ m &= 3 \end{aligned}$$

2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x .

Solution: We reduce one side to zero and express the other side as a polynomial in x . (Note that each coefficient is expressed as a polynomial in m .)

$$\begin{aligned} x^2 + 2mx + 46 &= 10x + 6m \\ x^2 + 2mx - 10x + 46 - 6m &= 0 \\ x^2 + (2m - 10)x + (-6m + 46) &= 0 \end{aligned}$$

We "pretend" that we know the value of m and complete the square. Half of the linear coefficient is $m - 5$ and so the complete square we will push our expression towards is

$$(x + (m - 5))^2 = x^2 + (2m - 10)x + (m - 5)^2$$

and so we smuggle in $(m - 5)^2$.

$$\begin{aligned} x^2 + (2m - 10)x + (-6m + 46) &= 0 \\ \underbrace{x^2 + (2m - 10)x + (m - 5)^2} - (m - 5)^2 + (-6m + 46) &= 0 \\ (x + (m - 5))^2 - (m - 5)^2 + (-6m + 46) &= 0 \\ (x + (m - 5))^2 - (m^2 - 10m + 25) + (-6m + 46) &= 0 \\ (x + (m - 5))^2 - m^2 + 10m - 25 - 6m + 46 &= 0 \\ (x + (m - 5))^2 - m^2 + 4m + 21 &= 0 \end{aligned}$$

For all values of m , this expression is quadratic. If we look at this as a parabola, its vertex has

$$\begin{aligned}x\text{-coordinate:} & \quad -(m - 5) \\y\text{-coordinate:} & \quad -m^2 + 4m + 21\end{aligned}$$

for exactly one solution (or x -intercept), the vertex needs to be on the x -axis. Which is the same as

$$\begin{aligned}-m^2 + 4m + 21 &= 0 \\-(m^2 - 4m - 21) &= 0 \\-(m - 7)(m + 3) &= 0 \implies m_1 = 7 \quad m_2 = -3\end{aligned}$$

Thus the equation will have one solution (and it will be $-(m - 5) = -m + 5$) when m is 7 or -3 . We can check: if $m = 7$, then

$$\begin{aligned}x^2 + 2mx + 46 &= 10x + 6m \quad \text{and } m = 7 \\x^2 + 2(7)x + 46 &= 10x + 6(7) \\x^2 + 14x + 46 &= 10x + 42 \\x^2 + 4x + 4 &= 0 \\(x + 2)^2 &= 0 \\x &= -2\end{aligned}$$

If $m = -3$, then

$$\begin{aligned}x^2 + 2mx + 46 &= 10x + 6m \quad \text{and } m = -3 \\x^2 + 2(-3)x + 46 &= 10x + 6(-3) \\x^2 - 6x + 46 &= 10x - 18 \\x^2 + 16x + 64 &= 0 \\(x + 8)^2 &= 0 \\x &= -8\end{aligned}$$

3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 9x + 7$ with slope -3 .

Solution: We are looking for the equation of a line with slope -3 . Its slope-intercept form is then $y = -3x + b$. We need to find the value of b .

$$\begin{aligned}y &= -3x + b \\y &= x^2 - 9x + 7\end{aligned}$$

We need to find the value(s) of b that will result in exactly one solution for the system shown above.

$$\begin{aligned}x^2 - 9x + 7 &= -3x + b \\x^2 - 6x + 7 - b &= 0 \\x^2 - 6x + 9 - 9 + 7 - b &= 0 \\(x - 3)^2 - 2 - b &= 0\end{aligned}$$

This quadratic equation has exactly one solution if the quantity added to the complete square is zero.

$$\begin{aligned}-2 - b &= 0 \\b &= -2\end{aligned}$$

4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0, 1)$.

Solution: We are looking for the equation of a line. Specifically, we will find its slope-intercept form, $y = mx + b$. Thus, we need to find the values of m and b . Clearly, $b = 1$ since the point given is the y -intercept.

$$\begin{aligned} y &= mx + b & (0, 1) \text{ is on the line} \\ 1 &= m(0) + b \\ 1 &= b & \implies y = mx + 1 \end{aligned}$$

Thus our tangent line, $y = mx + 1$. This equation represents all non-vertical lines passing through $(0, 1)$. We will obtain the value(s) of m using the fact that it must be a tangent line. The common points of the line and the parabola have coordinates that are solution of the system

$$\begin{aligned} y &= mx + 1 \\ y &= \frac{1}{2}x^2 + 3x + 3 \end{aligned}$$

If we use substitution to solve the system, we obtain the equation

$$mx + 1 = \frac{1}{2}x^2 + 3x + 3$$

Once we solve this quadratic equation, we will have the x -coordinates of all intersection points. Our line is a tangent line if this quadratic equation has exactly one solution.

$$\begin{aligned} mx + 1 &= \frac{1}{2}x^2 + 3x + 3 \\ 0 &= \frac{1}{2}x^2 - mx + 3x + 2 & \text{multiply by 2} \\ 0 &= x^2 - 2mx + 6x + 4 \\ 0 &= x^2 + (-2m + 6)x + 4 \end{aligned}$$

We complete the square: half of the linear coefficient is $-m + 3$

$$\begin{aligned} 0 &= x^2 + (-2m + 6)x + 4 \\ 0 &= \underbrace{x^2 + (-2m + 6)x + (-m + 3)^2}_{(x + (-m + 3))^2} - (-m + 3)^2 + 4 \\ 0 &= (x + (-m + 3))^2 - (-m + 3)^2 + 4 \\ 0 &= (x - m + 3)^2 - (-m + 3)^2 + 4 \end{aligned}$$

This expression represents a parabola with vertex

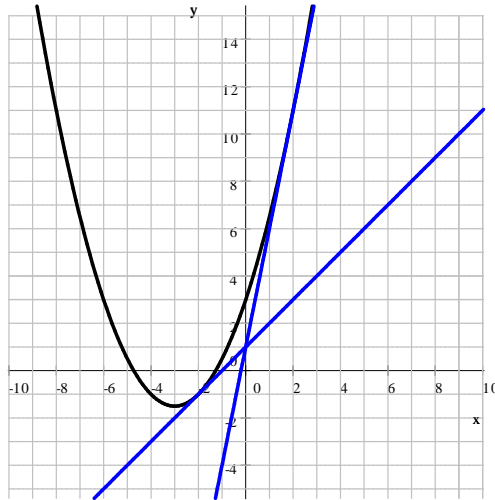
$$\begin{aligned} x\text{-coordinate:} & \quad m - 3 \\ y\text{-coordinate:} & \quad -(-m + 3)^2 + 4 \end{aligned}$$

There will be exactly one x -intercept if $-(-m + 3)^2 + 4 = 0$.

$$\begin{aligned} -(-m + 3)^2 + 4 &= 0 \\ 4 - (-m + 3)^2 &= 0 & \text{difference of squares theorem} \\ 2^2 - (-m + 3)^2 &= 0 \\ (2 + (-m + 3))(2 - (-m + 3)) &= 0 \\ (-m + 5)(m - 1) &= 0 & \implies m_1 = 5 \quad m_2 = 1 \end{aligned}$$

Thus the two tangent lines drawn from $(0, 1)$ to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ are

$$y = x + 1 \quad \text{and} \quad y = 5x + 1$$



5. Find an equation for all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 - 6x + 30$ from the point $(7, 8)$.

Solution: We start with the equation of all (except vertical) lines passing through $(7, 8)$. These lines can all be expressed using the slope-point form and then solving for y .

$$\begin{aligned} y - 8 &= m(x - 7) \\ y &= mx - 7m + 8 \end{aligned}$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{aligned} y &= \frac{1}{2}x^2 - 6x + 30 \\ y &= mx - 7m + 8 \end{aligned}$$

We want to find m so that this system has exactly one solution. We pretend that we know the value of m and compute for the points of intersection. Using substitution, we obtain the following quadratic (in x) equation.

$$\frac{1}{2}x^2 - 6x + 30 = mx - 7m + 8$$

We reduce one side to zero and arrange the terms as a polynomial in x .

$$\begin{aligned} \frac{1}{2}x^2 - 6x + 30 &= mx - 7m + 8 && \text{multiply by 2} \\ x^2 - 12x + 60 &= 2mx - 14m + 16 \\ x^2 - 12x + 60 - 2mx + 14m - 16 &= 0 \\ x^2 + (-2m - 12)x + 14m + 44 &= 0 \end{aligned}$$

We complete the square. Half of the linear coefficient is $\frac{-2m-12}{2} = -m-6$ and so we smuggle in $(-m-6)^2$ to obtain the complete square $(x + (-m-6))^2$

$$\underbrace{x^2 + (-2m-12)x + (-m-6)^2}_{(x + (-m-6))^2} - (-m-6)^2 + 14m + 44 = 0$$

$$(x + (-m-6))^2 - (-m-6)^2 + 14m + 44 = 0$$

For this quadratic equation to have exactly one solution, the quantity added to the complete square must be zero.

$$(x + (-m-6))^2 - \underbrace{(-m-6)^2 + 14m + 44}_{\text{needs to be zero}} = 0$$

This is a quadratic equation we can solve for m .

$$\begin{aligned} -(-m-6)^2 + 14m + 44 &= 0 \\ (-m-6)^2 - 14m - 44 &= 0 \\ m^2 + 12m + 36 - 14m - 44 &= 0 \\ m^2 - 2m - 8 &= 0 \\ (m-4)(m+2) &= 0 \implies m_1 = 4 \quad m_2 = -2 \end{aligned}$$

We substitute these values into the equation of the line $y - 8 = m(x - 7)$ to get the two equations

$$y - 8 = 4(x - 7) \quad \text{and} \quad y - 8 = -2(x - 7)$$

We can bring these to the slope intercept form and get

$$y = 4x - 20 \quad \text{and} \quad y = -2x + 22$$

6. Find an equation for all tangent lines drawn to the graph of $y = x^2 - 10x + 16$ from the point $(7, -14)$.

Solution: We start with the equation of all (except vertical) lines passing through $(7, -14)$. These lines can all be expressed using the slope-point form and then solving for y .

$$\begin{aligned} y + 14 &= m(x - 7) \\ y &= mx - 7m - 14 \end{aligned}$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{aligned} y &= x^2 - 10x + 16 \\ y &= mx - 7m - 14 \end{aligned}$$

We want to find m so that the system has exactly one solution. We pretend we know m and compute for the points of intersection.

$$\begin{aligned} x^2 - 10x + 16 &= mx - 7m - 14 \\ x^2 - 10x + 16 - mx + 7m + 14 &= 0 \\ x^2 - 10x - mx + 7m + 30 &= 0 \\ x^2 - (m+10)x + 7m + 30 &= 0 \end{aligned}$$

We complete the square.

$$\left(x - \frac{m+10}{2}\right)^2 = x^2 - (m+10)x + \frac{(m+10)^2}{4} \quad \text{So we smuggle in } \frac{(m+10)^2}{4}$$

$$\begin{aligned} x^2 - (m+10)x + \frac{(m+10)^2}{4} - \frac{(m+10)^2}{4} + 7m + 30 &= 0 \\ \left(x - \frac{m+10}{2}\right)^2 - \underbrace{\frac{(m+10)^2}{4} + 7m + 30}_{\text{has to be zero}} &= 0 \end{aligned}$$

Now we need to find those values of m for which this equation has exactly one solution.

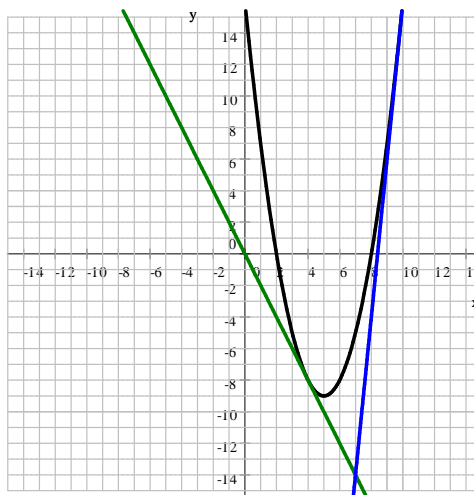
$$\begin{aligned} -\frac{(m+10)^2}{4} + 7m + 30 &= 0 && \text{multiply by } -4 \\ (m+10)^2 - 28m - 120 &= 0 \\ m^2 + 20m + 100 - 28m - 120 &= 0 \\ m^2 - 8m - 20 &= 0 \\ (m+2)(m-10) &= 0 \implies m_1 = -2 \quad m_2 = 10 \end{aligned}$$

We substitute these values into the equation of the line $y + 14 = m(x - 7)$ to get the two equations

$$y + 14 = -2(x - 7) \quad \text{and} \quad y + 14 = 10(x - 7)$$

We can bring these to the slope intercept form and get

$$y = -2x \quad (\text{green graph}) \quad \text{and} \quad y = 10x - 84 \quad (\text{blue graph})$$



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