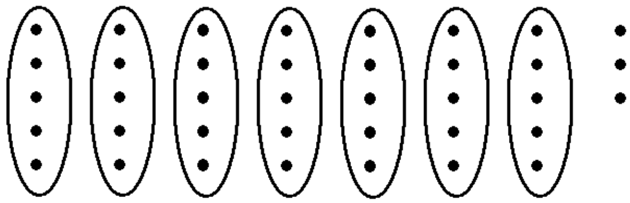
Theorem (Division with Remainder) For every integers N and m, $m \neq 0$, there exist unique integers q and r such that

$$N = mq + r \quad \text{and} \quad r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture below illustrates the division $38 \div 5 = 7$ R 3, or, in other form: $38 = 5 \cdot 7 + 3$.



Note that there are two different ways to represent this division: $38 \div 5 = 7$ R 3 and $\dfrac{38}{5} = 7\dfrac{3}{5}$

This theorem, also called the Euclidean division, is very fundamental to mathematics. There is a similar theorem about division of polynomials.

Theorem (Division with Remainder for Polynomials) For every polynomials $N(x)$ and $m(x)$, $m(x)$ not the constant zero polynomial, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$N(x) = m(x)\,q(x) + r(x) \quad \text{and the degree of } r(x) \text{ is less than that of } m(x)$$

A special case of this theorem is when we divide by a linear polynomial. By the theorem above, such a division results in a remainder that is a polynomial with a degree smaller than linar: that is, a constant polynomial.

Suppose that $N(x)$ is a polynomial and $m(x) = x - a$ is a linear polynomial. According to the theorem above, there exist a polynomial $q(x)$ and real number r such that

$$N(x) = q(x)(x - a) + r$$

The equality above is true between polynomial functions. Thus, the equality above is true for all values of x. Let $x = a$.

$$\begin{aligned} N(x) &= q(x)(x-a) + r \qquad \text{Let } a = x \\ N(a) &= q(a)(a-a) + r \\ N(a) &= q(a) \cdot 0 + r \\ N(a) &= r \end{aligned}$$

Theorem (Remainder Theorem) Suppose that $N(x)$ is a polynomial and $m(x) = x - a$. The remainder of the division of $N(x)$ by $x - a$ is equal to $N(a)$.

An immediate consequence of this theorem is when this remainder is zero.

Theorem: Suppose that $N(x)$ is a polynomial and a is a real number such that $N(a) = 0$. Then $x - a$ is a factor of $N(x)$, i.e. there eqists a polynomial $q(x)$ such that $N(x) = (x - a)\,q(x)$

Sample Problems

1. Use the remainder theorem to find the remainder in the division $\left(5x^4 - 8x^3 + 3x^2 - x - 1\right) \div (x - 2)$
2. Solve the equation $x^7 - 17x^5 + 36x^4 - 20x^3 = 0$ given that 2 is a solution of the equation.
3. Solve the equation $x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48 = 0$.

Practice Problems

1. Perform the indicated divisions with remainders:

   a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Use the remainder theorem to find the remainder in each of the following divisions.

   a) $\left(8x^5 - 2x^3 + x + 11\right) \div (x - 1)$

   b) $\left(8x^5 - 2x^3 + x + 11\right) \div (x + 1)$

   c) $\left(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2\right) \div (x - 2)$

   d) $\left(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2\right) \div (x + 2)$

3. Solve the each of the given equations.

   a) $x^4 + 7x^3 - 4x^2 - 28x = 0$ given that -2 is a solution

   b) $x^6 + x^5 - 18x^4 - 52x^3 - 40x^2 = 0$ given that 2 is a solution

   c) $x^4 + 8x^3 + 12x^2 - 32x - 64 = 0$ given that -4 is a solution

4. Solve each of the following equations.

   a) $x^6 - 15x^5 + 53x^4 - 21x^3 - 90x^2 = 0$

   b) $x^6 + 7x^5 - 47x^4 - 307x^3 - 394x^2 + 236x + 504 = 0$

Answers - Sample Problems

1. 25 2. $-5, 0, 1, 2$ 3. $-6, -1, 1, 8$

Answers - Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9 2. a) 18 b) 4 c) 84 d) 0

3. a) $-7, -2, 0, 2$ b) $-2, 0, 5$ c) $-4, -2, 2$ 4. a) $-1, 0, 3, 10$ b) $-9, -2, 1, 7$

Solutions - Sample Problems

1. Use the remainder theorem to find the remainder in the division $\left(5x^4 - 8x^3 + 3x^2 - x - 1\right) \div (x - 2)$

   Solution: Let $N(x) = 5x^4 - 8x^3 + 3x^2 - x - 1$. If we divide this polynomial by $m(x) = x - 2$, the remainder is the same as $N(a)$. Thus, we evaluate $N(x)$ at $x = 2$.

$$\begin{aligned} N(2) &= 5 \cdot 2^4 - 8 \cdot 2^3 + 3 \cdot 2^2 - 2 - 1 = 5 \cdot 16 - 8 \cdot 8 + 3 \cdot 4 - 2 - 1 \\ &= 80 - 64 + 12 - 2 - 1 = 25 \end{aligned}$$

   Thus the remainder is 25.

   b) $\left(x^5 + 3x^4 - x^3 - 2x^2 + x - 6\right) \div (x + 3)$

   Solution: Let $N(x) = x^5 + 3x^4 - x^3 - 2x^2 + x - 6$. If we divide this polynomial by $m(x) = x + 3$, the remainder is the same as $N(-3)$. Thus, we evaluate $N(x)$ at $x = -3$.

$$\begin{aligned} N(2) &= x^5 + 3x^4 - x^3 - 2x^2 + x - 6 = (-3)^5 + 3(-3)^4 - (-3)^3 - 2(-3)^2 + (-3) - 6 \\ &= -243 + 3 \cdot 81 - (-27) - 2 \cdot 9 - 3 - 6 = -243 + 243 + 27 - 18 - 3 - 6 = 0 \end{aligned}$$

   Thus the remainder is 0. This measn that $(x + 3)$ is a linear factor of $N(x)$. Indeed, if we perform the division, the quotient is $x^4 - x^2 + x - 2$ and the remainder is 0.

2. Solve the equation $x^7 - 17x^5 + 36x^4 - 20x^3 = 0$ given that 2 is a solution of the equation.

   Solution: To solve any equation of degree greater than one, we need to factor and apply the zero product rule. The greatest common factor is x^3. We factor it out:

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3\left(x^4 - 17x^2 + 36x - 20\right)$$

   We still need to factor $P(x) = x^4 - 17x^2 + 36x - 20$. Since $P(2) = 0$, $x - 2$ is a linear factor of $P(x)$. We divide $x^4 - 17x^2 + 36x - 20$ by $x - 2$. (For the steps to dividing a polynomials, please see the separate handout on dividing polynomials.) The quotient is $x^3 + 2x^2 - 13x + 10$. So far, we have that

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x - 2)\left(x^3 + 2x^2 - 13x + 10\right)$$

   We still need to factor $Q(x) = x^3 + 2x^2 - 13x + 10$. Let us substitute $x = 2$ into $Q(x)$.

$$Q(2) = 2^3 + 2 \cdot 2^2 - 13 \cdot 2 + 10 = 8 + 8 - 26 + 10 = 0$$

This means that $x-2$ is a factor of $Q(x)$. We divide: $x^3+2x^2-13x+10$ by $x-2$ and obtain the quotient x^2+4x-5 and the remainder is zero. So now we have

$$x^7-17x^5+36x^4-20x^3 = x^3(x-2)(x-2)\left(x^2+4x-5\right) = x^3(x-2)^2\left(x^2+4x-5\right)$$

We factor the quadratic expression x^2+4x-5 and obtain $(x+5)(x-1)$. Thus, we now have completely factored our polynomial:

$$\begin{aligned} x^7-17x^5+36x^4-20x^3 &= 0 \\ x^3(x-2)^2(x+5)(x-1) &= 0 \end{aligned}$$

$$x_1=0,\ \ x_2=2,\ \ x_3=-5,\ \ x_4=1$$

3. Solve the equation $x^6-2x^5-50x^4+4x^3+97x^2-2x-48=0$.

   Solution: Since this equation is of degree 6, we do not have obvious factoring techniques. So, we will rely on the remainder theorem. Let $P(x)=x^6-2x^5-50x^4+4x^3+97x^2-2x-48$. In order to find some factors, we will be looking for zeroes of the polynomial. We start by substituting easy, small numbers such as 1 , -1, 2, and -2. $x=1$ turns out to be a zero. So, $x-1$ is a factor. We divide:

$$\left(x^6-2x^5-50x^4+4x^3+97x^2-2x-48\right)\div(x-1) = x^5-x^4-51x^3-47x^2+50x+48$$

   So we have

$$P(x)=\left(x^5-x^4-51x^3-47x^2+50x+48\right)(x-1)$$

   We substitute $x=1$ into the quotient $x^5-x^4-51x^3-47x^2+50x+48$ and obtain zero. Thus, $x-1$ is still a factor. We divide

$$\left(x^5-x^4-51x^3-47x^2+50x+48\right)\div(x-1) = x^4-51x^2-98x-48$$

   So now we have that

$$P(x)=\left(x^4-51x^2-98x-48\right)(x-1)^2$$

   We substitute $x=1$ into the quotient $x^4-51x^2-98x-48$ and obtain -196. This means that we have exhausted $x=1$.

   Let us try $x=-1$. We evaluate $Q(x)=x^4-51x^2-98x-48$ at $x=-1$. Since the result is zero, $x+1$ is a factor of $Q(x)$. We divide

$$\left(x^4-51x^2-98x-48\right)\div(x+1) = x^3-x^2-50x-48$$

   So we now have that

$$P(x)=\left(x^3-x^2-50x-48\right)(x-1)^2(x+1)$$

   We evaluate $T(x)=x^3-x^2-50x-48$ at $x=-1$. Since the result is zero, $x+1$ is a factor of $T(x)$. We divide

$$\left(x^3-x^2-50x-48\right)\div(x+1) = x^2-2x-48$$

   So we now have that

$$P(x)=\left(x^2-2x-48\right)(x-1)^2(x+1)^2$$

   We can easily factor the quadratic factor: $x^2-2x-48=(x+6)(x-8)$. Now we have completely factored our polynomial.

$$\begin{aligned} x^6-2x^5-50x^4+4x^3+97x^2-2x-48 &= 0 \\ (x+6)(x-8)(x-1)^2(x+1)^2 &= 0 \end{aligned}$$

$$x_1=-6,\ x_2=8,\ x_3=1,\ x_4=-1$$

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

