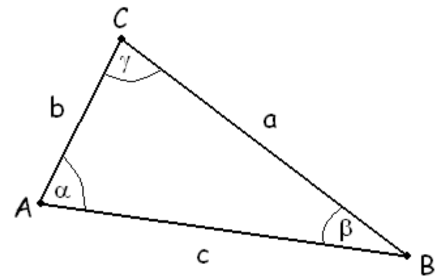
The Pythagorean theorem is a fundamental theorem about the connection between the three sides of a right triangle. Let us recall first a few things we will need for this topic.

Standard labeling simplifies notation. According to standard labeling, side a is opposite point A and angle α. Similarly, side b is opposite of point B and angle β, and side c is opposite point C and angle γ.

Also recall that the three angles in any triangle add up to $180°$.



In any triangle, the order between sides is the same as the order between the angles opposite them. For example, the longest side is always opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite of the shortest side. If two sides are equally long, the angles opposite them are also equal.

In a right triangle, the right angle is the single greatest angle in the triangle, because the other two angles must add up to $90°$ so they both are smaller than $90°$.

Therefore, right triangles have a single longest side and it is located opposite of the right angle.

Definition: In a right triangle, the side opposite the right angle is the longest side. We call this side the **hypotenuse** of the triangle.

We are now ready to state the Pythagorean theorem. It actually has two parts.

Theorem: (The Pythagorean theorem)

Part 1. If ABC is a right triangle with shorter sides a and b and hypotenuse c, then
$$a^2 + b^2 = c^2$$

Part 2. In any triangle with sides a, b, and c, if $a^2 + b^2 = c^2$, then the angle opposite side c measures $90°$.

The first part tells us how right triangles behave. The second part tells us that *only* right triangles have this behavior.

Example 1. In each case, determine whether the three sides given are sides in a right triangle or not.

a) $5\,\mathrm{cm}$, $7\,\mathrm{cm}$, and $9\,\mathrm{cm}$ b) $5\,\mathrm{ft}$, $13\,\mathrm{ft}$ and $12\,\mathrm{ft}$ c) 53 units, 28 units, and 45 units.

Solution: We will use the second part of the Pythagorean theorem. Let us denote the longest side by c and the other two sides by a and b. If the statement $a^2 + b^2 = c^2$, is true, the triangle is a right triangle. If not, the triangle has no right angle in it.

a) The only side here that could be the hypotenuse, is the longest one, measuring $9\,\mathrm{cm}$. Therefore, the two quantities we need to compare are $(5\,\mathrm{cm})^2 + (7\,\mathrm{cm})^2$ and $(9\,\mathrm{cm})^2$.

$(5\,\mathrm{cm})^2 + (7\,\mathrm{cm})^2 = 25\,\mathrm{cm}^2 + 49\,\mathrm{cm}^2 = 74\,\mathrm{cm}^2$ and $(9\,\mathrm{cm})^2 = 81\,\mathrm{cm}^2$

Since $74\,\mathrm{cm}^2 \neq 81\,\mathrm{cm}^2$, this triangle is $\boxed{\text{not a right triangle.}}$

b) The only side here that could be the hypotenuse, is the longest one, measuring $13\,\mathrm{ft}$. Therefore, the two quantities we need to compare are $(5\,\mathrm{ft})^2 + (12\,\mathrm{ft})^2$ and $(13\,\mathrm{ft})^2$.

$(5\,\mathrm{ft})^2 + (12\,\mathrm{ft})^2 = 25\,\mathrm{ft}^2 + 144\,\mathrm{ft}^2 = 169\,\mathrm{ft}^2$ and $(13\,\mathrm{ft})^2 = 169\,\mathrm{ft}^2$

Since $169\,\mathrm{ft}^2 = 169\,\mathrm{ft}^2$, this triangle is $\boxed{\text{a right triangle}}$ with hypotenuse $13\,\mathrm{ft}$ long.

c) The only side here that could be the hypotenuse, is the longest one, measuring 53 units. Therefore, the two quantities we need to compare are $28^2 + 45^2$ and 53^2.

$28^2 + 45^2 = 784 + 2025 = 2809$ and $53^2 = 2809$

Since $2809 = 2809$, this triangle is $\boxed{\text{a right triangle.}}$

If we know two sides of a right triangle, the Pythagorean theorem enables us to find the third side.

Example 2. In each case, find the missing side of the right triangle.

a) the shortest two sides are $12\,\mathrm{cm}$ and $35\,\mathrm{cm}$

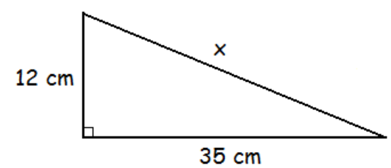
b) the longest two sides are $15\,\mathrm{ft}$ and $17\,\mathrm{ft}$

c) the shortest two sides are both 1 unit long

Solution: a) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x.

We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} 12^2 + 35^2 &= x^2 \\ 144 + 1225 &= x^2 \\ 1369 &= x^2 \\ \pm 37 &= x \end{aligned}$$

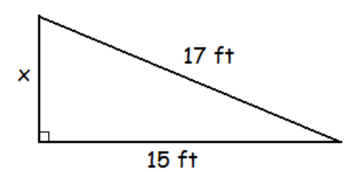


Since x represents a distance, and distances can never be negative, $x = -37$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is $\boxed{37\,\mathrm{cm}}$.

b) We will use the first part of the Pythagorean theorem. Let us denote the missing shortest side by x.

We state the Pythagorean theorem for this right triangle.

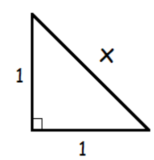
$$\begin{aligned} x^2 + 15^2 &= 17^2 \\ x^2 + 225 &= 289 \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$



Since distances cannot be negative, $x = -8$ is ruled out as a possible solution. Thus the missing side is $\boxed{8\,\mathrm{ft}}$ long.

c) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x.
We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} 1^2 + 1^2 &= x^2 \\ 1 + 1 &= x^2 \\ 2 &= x^2 \\ \pm\sqrt{2} &= x \end{aligned}$$



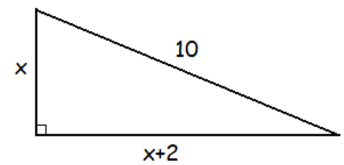
Since x represents a distance, and distances can never be negative, $x = -\sqrt{2}$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is $\boxed{\sqrt{2}\text{ unit}}$.

This last example shows that right triangles often have side lengths that are irrational numbers. This particular triangle is responsible for bringing $\sqrt{2}$ into the mathematics of the ancient Greeks. They did not have the concept of irrational numbers yet, and so couldn't really deal with $\sqrt{2}$.

In order to find missing sides, we do not always need to know the lengths of two sides if other information is given.

Example 3. The hypotenuse of a right triangle is 10 units long. The difference between the other two sides is 2 units. Find the missing sides of the right triangle.

Solution: If we label the shortest side by x, then the other missing side can be denoted by $x + 2$. We state the Pythagorean theorem for this right triangle and solve the equation for x.



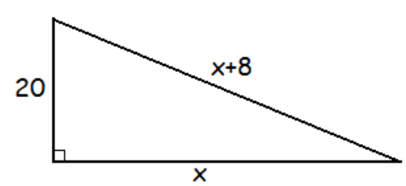
$$
\begin{aligned}
x^2 + (x+2)^2 &= 10^2 && \text{expand complete square}\\
x^2 + x^2 + 4x + 4 &= 10 && \text{combine like terms}\\
2x^2 + 4x + 4 &= 100 && \text{subtract 100}\\
2x^2 + 4x - 96 &= 0 && \text{factor out 2}\\
2\left(x^2 + 2x - 48\right) &= 0 && \text{factor (we wil complete the square)} \quad (x+1)^2 = x^2 + 2x + 1\\
2\left(\underbrace{x^2 + 2x + 1} - 1 - 48\right) &= 0\\
2\left((x+1)^2 - 49\right) &= 0\\
2\left((x+1)^2 - 7^2\right) &= 0\\
2(x+1+7)(x+1-7) &= 0\\
2(x+8)(x-6) &= 0 \quad \Longrightarrow \quad x_1 = -8 \ \ x_2 = 6
\end{aligned}
$$

Since distances can never be negative, -8 is ruled out as a possible solution. The other solution, $x = 6$ means that the other side, denoted by $x+2$ must be $6+2 = 8$ units long. Thus, this right triangle has sides $\boxed{\text{6, 8, and 10 units}}$ long. We check: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, thus the triangle is indeed right. Also, $8 - 6 = 2$, so our answer is correct.

The following example will be a similar application problem, but the computation will be entirely different.

Example 4. The shortest side of a right triangle is 20 units long. The difference between the other two sides is 8 units. Find the missing sides of the right triangle.

Solution: If we label the shorter missing side by x, then the other missing side, the hypotenuse can be denoted by $x + 8$. We state the Pythagorean theorem for this right triangle and solve the equation for x.



$$
\begin{aligned}
20^2 + x^2 &= (x+8)^2 && \text{expand complete square}\\
400 + x^2 &= x^2 + 16x + 64 && \text{subtract } x^2\\
400 &= 16x + 64 && \text{subtract 64}\\
336 &= 16x && \text{divide by 16}\\
21 &= x
\end{aligned}
$$

Therefore, the shoter missing side, denoted by x is 21 units long, and the hypotenuse, denoted by $x+8$ must be $21+8=29$ units long. So the three sides of this right triangle are $\boxed{\text{20, 21, and 29 units}}$ long. We check: the difference between 29 and 21 is indeed 8. For the Pythagorean theorem, $20^2+21^2=400+441=841$ and $29^2=841$, and so $20^2+21^2=29^2$. This means that this is indeed a right triangle, and so our answer is correct.

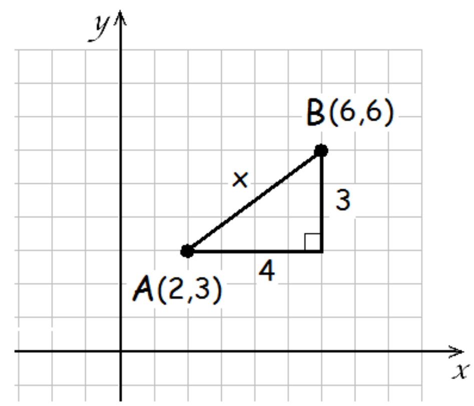
This problem turned out to be much easier because the equation was linear after x^2 was subtracted from both sides.

The following example is an extremely important application of the Pythagorean theorem.

Example 5. Find the distance between the given points.

a) $A(2,3)$ and $B(6,6)$ b) $P(-3,2)$ and $Q(1,4)$

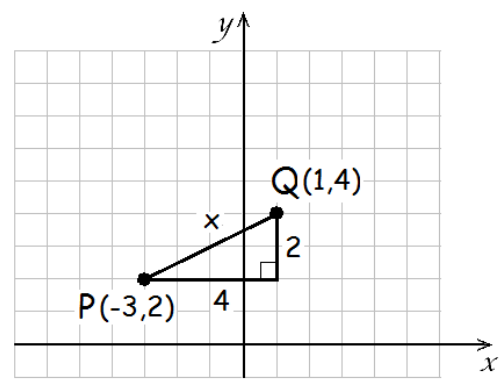
Solution: a) Let us plot the given points in a coordinate system as shown. The right triangle we created has shorter sides 4 and 3 units long. We denote the line segment connecting A and B (the hypotenuse) by x and state the Pythagorean theorem.



$$\begin{aligned} 4^2+3^2 &= x^2 \\ 16+9 &= x^2 \\ 25 &= x^2 \\ \pm 5 &= x \end{aligned}$$

Since distances can never be negative, -5 is ruled out and so our answer is $\boxed{\text{5 units}}$.

b) Let us plot the given points in a coordinate system as shown. The right triangle has shorter sides 2 and 4 units long. We denote the line segment connecting P and Q (the hypotenuse) by x and state the Pythagorean theorem for the triangle.



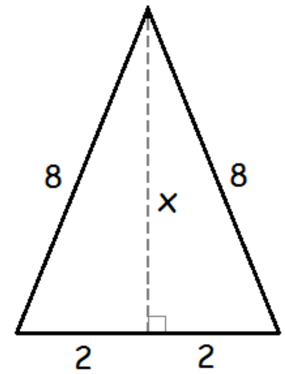
$$\begin{aligned} 4^2+2^2 &= x^2 \\ 16+4 &= x^2 \\ 20 &= x^2 \\ \pm\sqrt{20} &= x \end{aligned}$$

Since distances can never be negative, $-\sqrt{20}$ is ruled out and so our answer is $\sqrt{20}$ units. We might simplify $\sqrt{20}$ and write it as $2\sqrt{5}$. Then our final answer is $\boxed{2\sqrt{5}\text{ units}}$.

If the two points are $A(x_1,y_1)$ and $B(x_2,y_2)$, then the horizontal side of the triangle is the difference between the $x-$ coordinates of the points, and can be expressed as x_2-x_1. Similarly, the vertical side of the triangle is the difference between the $y-$ coordinates of the points, and can be expressed as y_2-y_1. If we pack all the steps in a single formula, **we get the distance formula**:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example 6. The sides of an isosceles triangle are 4 units, 8 units, and 8 units long. Find the exact value of the length of the height drawn to the 4 units long side.



Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles. The height now can be easily computed via the Pythagorean theorem applied to one of these right triangles. It is important to recognize that the missing side in the right triangle is not the hypotenuse.

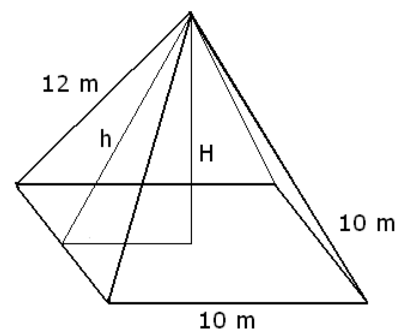
$$\begin{aligned}
2^2 + x^2 &= 8^2 \\
4 + x^2 &= 64 \\
x^2 &= 60 \\
x &= \pm\sqrt{60} = \pm 2\sqrt{15} \quad \Longrightarrow \quad x = 2\sqrt{15}
\end{aligned}$$

Again, the negative solution of the equation is ruled out because distances can not be negative. The height belonging to the base is $2\sqrt{15}$ units long. Because we are asked for the exact value,we must present our answer in radical form, as $\boxed{2\sqrt{15}\text{ units}}$.

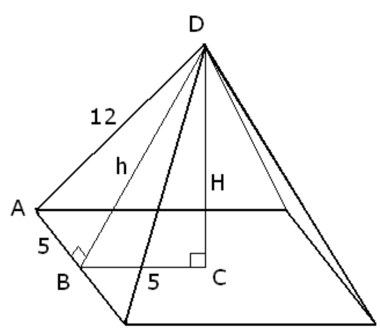
Example 7. Consider the square-based pyramid shown on the picture. Its base has sides $10\,\mathrm{m}$ (meters) long. The other faces of the pyramid are isosceles triangles with sides $10\,\mathrm{m}$, $12\,\mathrm{m}$, and $12\,\mathrm{m}$ long.

a) Find the exact value of h, the length of the height in a triangular face.

b) Find the exact value of H, the length of the height of the pyramid.



Solution: a) Let us label some of the vertices as shown on the figure below. We can find the value of h by stating the Pythagorean theorem on the right triangle ABD. The hypotenuse is the side opposite the right angle.



$$\begin{aligned}
(5\,\mathrm{m})^2 + h^2 &= (12\,\mathrm{m})^2 \\
25\,\mathrm{m}^2 + h^2 &= 144\,\mathrm{m}^2 \\
h^2 &= 119\,\mathrm{m}^2 \\
h &= \pm\sqrt{119\,\mathrm{m}^2} \quad \Longrightarrow \quad h = \sqrt{119}\,\mathrm{m}
\end{aligned}$$

Therefore, the height of the triangular face is $\boxed{\sqrt{119}\,\mathrm{m}}$. This is also called **the slant height** of the pyramid.

b) Let use the labels shown on the figure above. We can find the value of H by stating the Pythagorean theorem on the right triangle BCD. The hypotenuse is the side opposite the right angle.

$$\begin{aligned}
(5\,\mathrm{m})^2 + H^2 &= h^2 \\
(5\,\mathrm{m})^2 + H^2 &= \left(\sqrt{119}\,\mathrm{m}\right)^2 \\
25\,\mathrm{m}^2 + H^2 &= 119\,\mathrm{m}^2 \\
H^2 &= 94\,\mathrm{m}^2 \\
H &= \pm\sqrt{94\,\mathrm{m}^2} \quad \Longrightarrow \quad H = \sqrt{94}\,\mathrm{m}
\end{aligned}$$

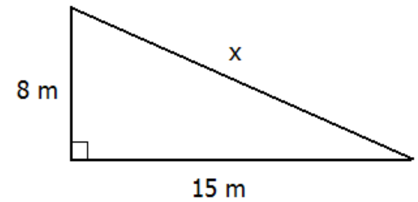
Thus the height of the pyramid is $\boxed{\sqrt{94}\,\mathrm{m}}$ long.

Sample Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

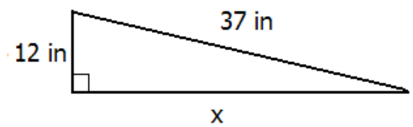
   a) $6\,\mathrm{cm}$, $10\,\mathrm{cm}$, and $8\,\mathrm{cm}$ b) $7\,\mathrm{ft}$, $15\,\mathrm{ft}$, and $50\,\mathrm{ft}$ c) $4\,\mathrm{m}$, $5\,\mathrm{m}$, and $6\,\mathrm{m}$

2. Find the hypotenuse of the triangle shown on the figure.



3. Find the missing leg of the right triangle shown on the picture.

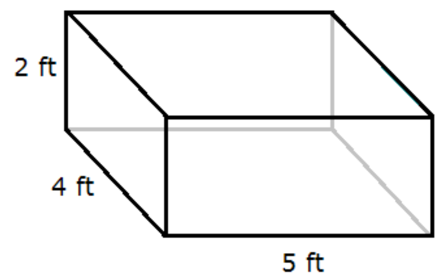
4. Find the distance between the points $(3,8)$ and $(8,-4)$.



5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

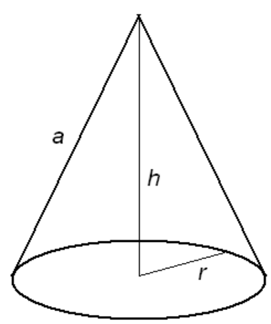
6. The hypotenuse of a right triangle is $20\,\mathrm{cm}$. The difference between the other two sides is $4\,\mathrm{cm}$. Find the sides of the triangle.

7. Find the length of the longest line segment inside the rectangular prism shown on the picture. (This line segment is called the **main diagonal** of the prism.)

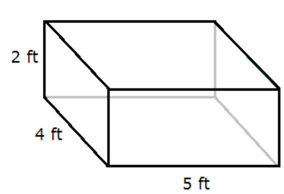


8. Find the height h of the cone shown on the picture, if the base has a radius of $r = 10\,\mathrm{m}$ and $a = 26\,\mathrm{m}$.

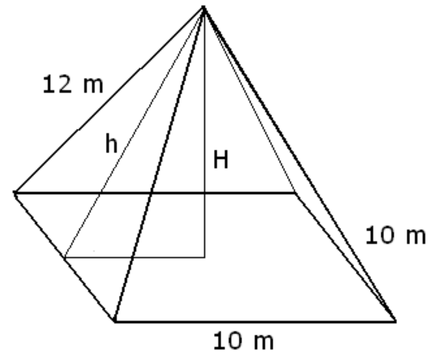
9. An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.



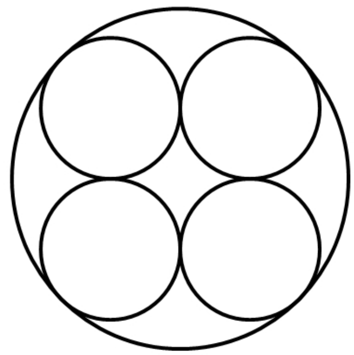
10. Find the length of the longest line segment (called the main diagonal) in the rectangular prism shown on the picture.



11. A pyramid (shown on the picture) has a square base with sides $10\,\mathrm{m}$ (meters) long. The other faces of the pyramid are isosceles triangles with sides $10\,\mathrm{m}$, $12\,\mathrm{m}$, and $12\,\mathrm{m}$.
a) Find the exact value of h, the length of the height in a triangular face.
b) Find the exact value of H, the length of the height of the pyramid.



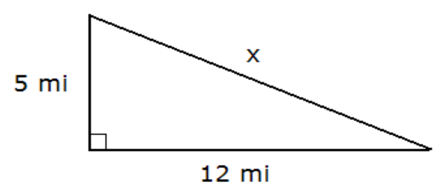
12. Four identical circles are arranged in a fifth circle as shown on the picture below. Compute the exact value of the radius of the big circle if the smaller circles have radius 1.



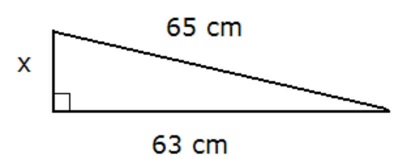
Practice Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.
a) $3\ \mathrm{cm}, 7\ \mathrm{cm}$, and $8\,\mathrm{cm}$ b) $37\,\mathrm{ft}$, $12\,\mathrm{ft}$, and $35\,\mathrm{ft}$ c) $6\,\mathrm{m}$, $7\,\mathrm{m}$, and $8\,\mathrm{m}$

2. Find the hypotenuse of the triangle shown on the figure.



3. Find the missing leg of the right triangle shown on the picture.



4. Find the length of the diagonal in a rectangle with sides 20 ft and 21 ft long.

5. Find the length of the diagonal of a square with sides 1 unit long.

6. The sides of an isosceles triangle are $25\,\mathrm{m}$, $25\,\mathrm{m}$, and $14\,\mathrm{m}$ long. Find the length of the height drawn to the $14\,\mathrm{m}$ long side.

7. Two sides of a right triangle are $8\,\mathrm{cm}$ and $17\,\mathrm{cm}$ long. Find the length of the missing side.

8. Find the distance between the given points. a) $(-2,-3)$ and $(6,3)$ b) $(-9,-3)$ and $(15,4)$.

9. One leg of a right triangle is $9\,\mathrm{cm}$. The difference between the other two sides is $1\,\mathrm{cm}$. Find the length of all sides.

10. The hypotenuse of a right triangle is 50 in. The difference between the other two sides is 34 in. Find the length of all sides.

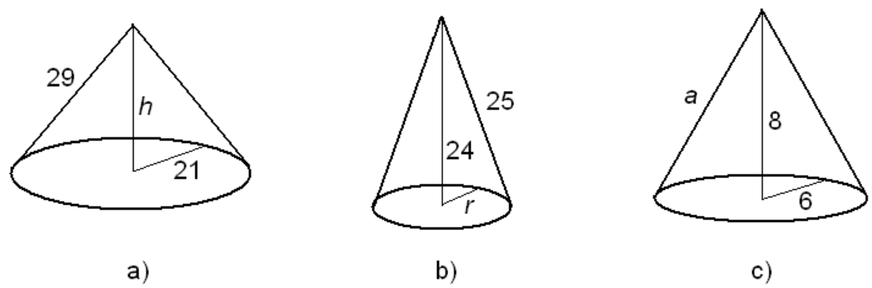
11. The shortest side of a right triangle is 16 units long. The difference between the other two sides is 4 units. Find the sides of this right riangle.

12. Consider a triangle with sides 7 unit, 7 unit, and 12 units long.

    a) Compute the exact value of the height drawn to the longest side.

    b) Compute the exact value of the area of the triangle.

13. Find the missing lengths indicated on the picture below.



14. Find the length of the main diagonal in a rectangular prism with sides

    a) $2\,\mathrm{m}$, $10\,\mathrm{m}$, and $11\,\mathrm{m}$ b) $5\,\mathrm{ft}$, $7\,\mathrm{ft}$, and $1\,\mathrm{ft}$

15. The height of a regular (same as equilateral) triangle is 6 feet. How long are its sides? Give the exact value of the answer.

16. The diagonal of a square is 80 meters long. How long are the sides?

17. Suppose that C is a center of a circle and P is a point (outside of the circle) 9 units away from C. We draw a tangent line to the circle from point P. Let Q be the point of tangency. Given that line segment PQ is 8 units long, compute the exact value of

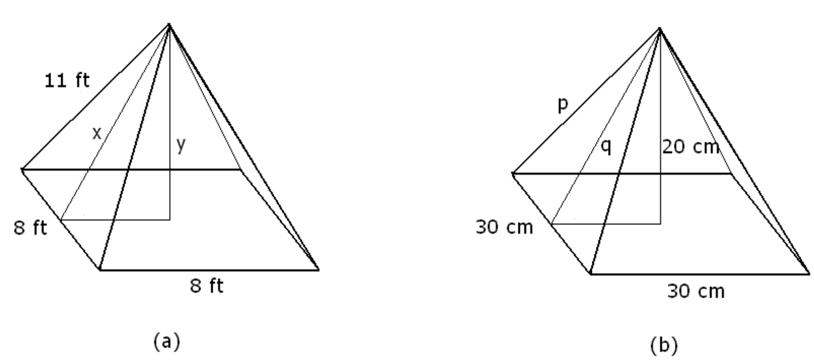
    a) the radius of the circle. b) the area of triangle CPQ.

18. An arch is in the shape of a semicircle. At a point along the base 2 meters from an end of the arch, the height of the arch is 10 meters. Find the maximum height of the arch.

19. How long is the main diagonal in a cube of sides 1 meter long?

20. Find the exact value of the missing lengths, labeled

    a) x and y b) p and q

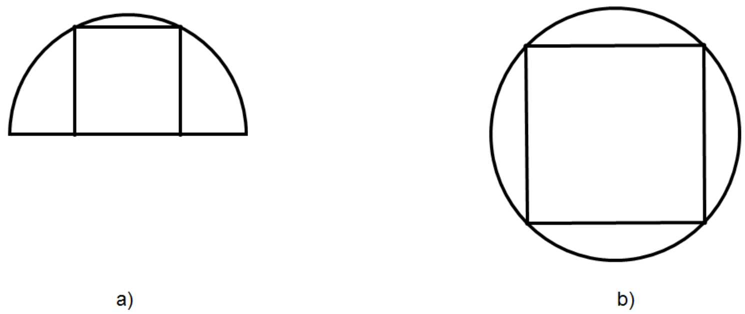


21. Consider a pyramid with a square base. The base has sides each 20 meters long. All other edges are 20 meters long.

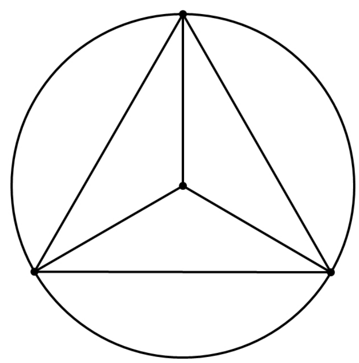
    a) Compute the exact value of the height of the pyramid.

    b) The volume of a pyramid can be computed as $V = \frac{1}{3}Bh$ where B is the area of its base and h is its height. Compute the volume of this pyramid. Give both the exact value and an approximation, accurate up to four or more digits after the decimal point.

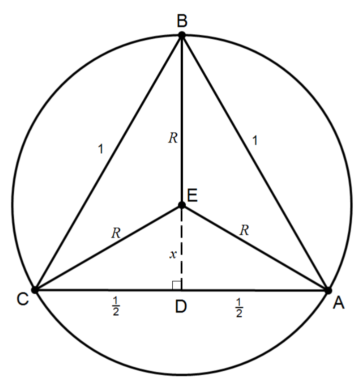
22. a) Compute the area of a square written into a semicirle with radius 1.
    b) Compute the area of a square written into a cirle with radius 1.



23. Consider a regular triangle width sides 1 unit long, written into a circle as shown on the picture below.
    a) Find the exact value of the radius of the circle.
    b) The center of the circle splits the height of the traingle into two parts. What is the ratio of the shorter part to the longer part?



Hint: Consider the picture shown below. State the Pythagorean theorem on triangles BCD and CDE and solve the system.



Answers

Sample Problems

1. a) There is a right angle opposite the $10\,\mathrm{cm}$ long side. b) not a right triangle 2. $17\,\mathrm{m}$ 3. $35\,\mathrm{in}$

4. 13 units 5. 20 units 6. $32\,\mathrm{cm}$ and $60\,\mathrm{cm}$ 7. $\sqrt{45}\,\mathrm{ft}$ 8. $24\,\mathrm{m}$ 9. $25\,\mathrm{ft}$ 10. $\sqrt{45}\,\mathrm{ft}$

11. $h = \sqrt{119}\,\mathrm{m}$ $H = \sqrt{94}\,\mathrm{m}$ 12. $1 + \sqrt{2}$

Practice Problems

1. a) not a right triangle b) There is a right angle opposite the $37\,\mathrm{ft}$ long side. c) not a right triangle

2. $13\,\mathrm{mi}$ 3. $16\,\mathrm{cm}$ 4. $29\,\mathrm{ft}$ 5. $\sqrt{2}$ unit 6. $24\,\mathrm{m}$ 7. $15\,\mathrm{cm}$ or $\sqrt{353}\,\mathrm{cm}$

8. a) 10 units b) 25 units 9. $9\,\mathrm{cm}$, $40\,\mathrm{cm}$, and $41\,\mathrm{cm}$ 10. $14\,\mathrm{in}$, $48\,\mathrm{in}$, and $50\,\mathrm{in}$

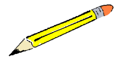
11. 16, 30, and 34 units 12. a) $\sqrt{13}$ unit b) $6\sqrt{13}$ unit^2 13. a) 20 b) 7 c) 10

14. a) $15\,\mathrm{m}$ b) $\sqrt{75}\,\mathrm{ft}$ 15. $4\sqrt{3}\,\mathrm{ft}$ 16. $40\sqrt{2}\,\mathrm{m}$ 17. a) $\sqrt{17}$ unit b) $4\sqrt{17}$ unit^2

18. $26\,\mathrm{m}$ 19. $\sqrt{3}\,\mathrm{m}$ 20. a) $x = \sqrt{105}\,\mathrm{ft}$ $y = \sqrt{89}\,\mathrm{ft}$ b) $q = 25\,\mathrm{cm}$, $p = \sqrt{850}\,\mathrm{cm} = 5\sqrt{34}\,\mathrm{cm}$

21. a) $10\sqrt{2}\,\mathrm{m}$ b) $\dfrac{4000}{3}\sqrt{2}\,\mathrm{m}^3 \approx 1885.618\,08\,\mathrm{m}^3$ 22. a) $\dfrac{4}{5}$ unit^2 b) 2 unit^2

23. a) $R = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ b) 1 to 2

Sample Problems  Solutions

1. Could the three line segments given be the three sides of a right triangle? Explain your answer.

   a) $6\ \mathrm{cm}$, $10\ \mathrm{cm}$, and $8\ \mathrm{cm}$

   Solution: The longest side is $10\ \mathrm{cm}$ long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

$$6^2 + 8^2 \stackrel{?}{=} 10^2$$

   We get that the two quantities are equal, thus this triangle has a right angle.

   b) $4\ \mathrm{m}$, $5\ \mathrm{m}$, and $6\ \mathrm{m}$

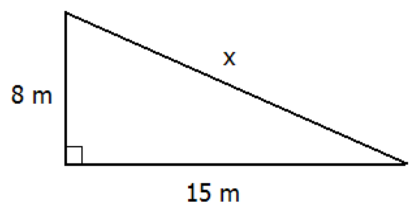
   Solution: The longest side is $6\ \mathrm{m}$ long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

$$4^2 + 5^2 \stackrel{?}{=} 6^2$$

$$\begin{aligned} \text{LHS} &= 4^2 + 5^2 = 16 + 25 = 41 \\ \text{RHS} &= 6^2 = 36 \qquad\qquad \text{LHS} \neq \text{RHS} \end{aligned}$$

   We get that the two quantities are not equal, thus this triangle does not have a right angle.

2. Find the hypotenuse of the triangle shown on the figure.



   Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned} 8^2 + 15^2 &= x^2 \\ 289 &= x^2 \\ \pm 17 &= x \end{aligned}$$

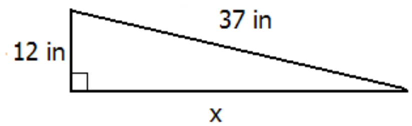
   Since distance can not be negative, -17 is ruled out. The answer is $17\,\mathrm{m}$.

   Please note that the step taking us from $x^2 = 289$ to $x = \pm 17$ is a very nice shortcut. The tradiotional way of solving quadratic equations is to reduce one side to zero, factor, and apply the zero product rule.

$$\begin{aligned} x^2 &= 289 \\ x^2 - 289 &= 0 \\ x^2 - 17^2 &= 0 \\ (x+17)(x-17) &= 0 \qquad \Longrightarrow \qquad x = -17 \text{ or } x = 17 \end{aligned}$$

   Students are encouraged to use the shorter version, **as long as they don't make the serious algebraic error** of concluding from $x^2 = 289$ that $x = 17$. While in the context of the geometry the negative solution is not possible, the equation $x^2 = 289$ has two solutions, 17 and -17.

3. Find the missing leg of the right triangle shown on the picture.



Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned} (12\,\mathrm{in})^2 + x^2 &= (37\,\mathrm{in})^2 \\ x^2 + 144\,\mathrm{in}^2 &= 1369\,\mathrm{in}^2 && \text{subtract } 144\,\mathrm{in}^2 \\ x^2 &= 1225\,\mathrm{in}^2 && \sqrt{1225} = 35 \\ x &= \pm 35\,\mathrm{in} \end{aligned}$$

Since distance can not be negative, $-35\,\mathrm{in}$ is ruled out. The answer is 35 inches.

4. Find the distance between the points $(3, 8)$ and $(8, -4)$.

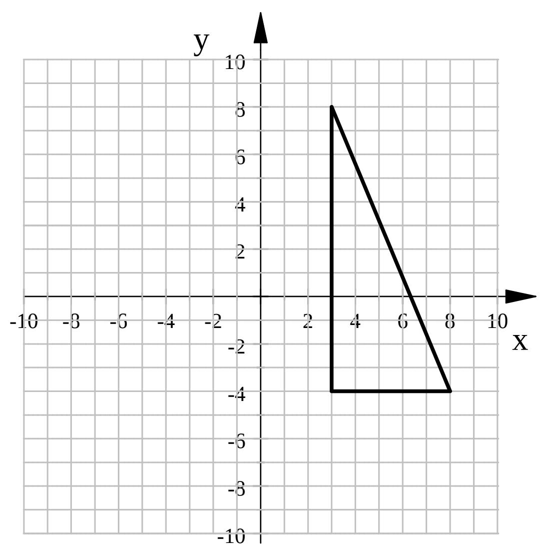
Solution: We graph the points, and draw a horizontal and vertical line connecting the points as shown on the picture. We can compute the distance as the hypotenuse of the right triangle we created. How long are the shorter sides?

Algebraic approach: Subtract the coordinates. The length of the horizontal side is the difference between the $x-$coordinates: $8 - 3 = 5$ and the length of the vertical side is the difference between the $y-$coordinates: $8 - (-4) = 12$.

The difference will alwys work. Even if we get -5 instead of 5, it will not matter since we will square it in the Pythagorean theorem.

Geometric approach: From 3 to 8 we have to step 5 units up. From -4 to 8 : first we step 4 to get from -4 to 0. Then another 8 steps to 8, and so $4 + 8 = 12$ steps. The message here is that the algebra and geometry will always agree.

Now we know that the shorter sides are 5 and 12 units long, and we need to find the hypotenuse.



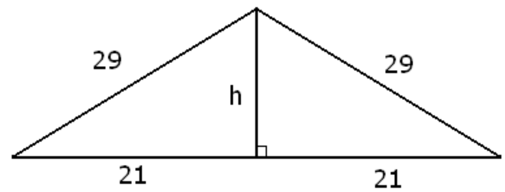
$$\begin{aligned} 5^2 + 12^2 &= x^2 \\ 25 + 144 &= x^2 \\ 169 &= x^2 \\ \pm 13 &= x \end{aligned}$$

Since distances are never negative, -13 is ruled out and so the answer is 13 units.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

   Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles as shown on the picture. The height now can be easily computed via the Pythagorean theorem.

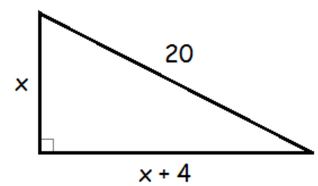
$$\begin{aligned} 21^2 + h^2 &= 29^2 \\ 441 + h^2 &= 841 \\ h^2 &= 400 \\ h &= \pm 20 \implies h = 20 \end{aligned}$$



   Again, the negative solution of the equation is ruled out because distances cannot be negative. The height belonging to the base is 20 units long.

6. The hypotenuse of a right triangle is 20 cm. The difference between the other two sides is 4 cm. Find the sides of the triangle.

   Solution: Let x denote the shortest side. Then the other missing side is $x + 4$ cm long. We state the Pythagorean theorem for the triangle and solve the quadratic equation for x.



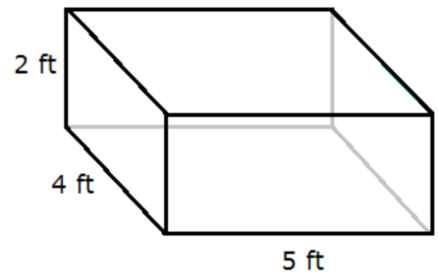
$$\begin{aligned} x^2 + (x+4)^2 &= 20^2 && \text{expand } (x+4)^2 \\ x^2 + x^2 + 8x + 16 &= 400 && \text{combine like terms} \\ 2x^2 + 8x + 16 &= 400 && \text{subtract 400} \\ 2x^2 + 8x - 384 &= 0 && \text{factor out 2} \\ 2\left(x^2 + 4x - 192\right) &= 0 && (x+2)^2 = x^2 + 4x = 4 \\ 2\left(\underbrace{x^2 + 4x + 4} - 4 - 192\right) &= 0 \\ 2\left((x+2)^2 - 196\right) &= 0 \\ 2\left((x+2)^2 - 14^2\right) &= 0 \\ 2(x+2+14)(x+2-14) &= 0 \\ 2(x+16)(x-12) &= 0 \end{aligned}$$

$$x_1 = -16, \ x_2 = 12$$

   Since distances are never negative, -16 is ruled out. If the shortest side is 12 cm, the other side is 12 cm + 4 cm = 16 cm. Thus the solution is 12 cm and 16 cm. We check:

$$16 - 12 = 4 \checkmark \text{ and } 16^2 + 12^2 = 256 + 144 = 400 = 20^2 \checkmark$$

7. Find the length of the longest line segment inside the rectangular prism shown on the picture. (This line segment is called the **main diagonal** of the prism.)

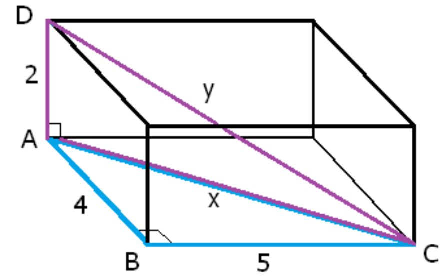


Solution: We will apply the Pythagorean theorem twice.

Let us label the points and sides we will use on the picture first.

We will find x using the Pythagorean theorem in triangle ABC.

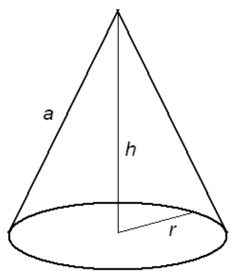
Then we can find y using the Pythagorean theorem in triangle ACD.



$$\begin{aligned} 4^2+5^2 &= x^2 \\ 41 &= x^2 \quad \Longrightarrow x=\sqrt{41} \end{aligned} \qquad \begin{aligned} 2^2+x^2 &= y^2 \\ 45 &= y^2 \quad \Longrightarrow y=\sqrt{45} \end{aligned}$$

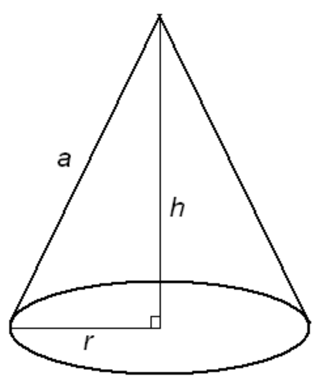
Note: Our result is actually $\sqrt{2^2+4^2+5^2}$. Indeed, we can see that the length of the main diagonal in a rectangular prism.with sides x, y, and z is $L=\sqrt{x^2+y^2+z^2}$. This is sometimes called the 3-dimensional Pythagorean theorem.

8. Find the height h of the cone shown on the picture, if the base has a radius of $10\,\mathrm{m}$ and $a=26\,\mathrm{m}$.



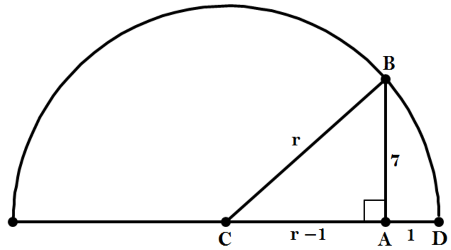
Solution: There is a right triangle formed by a, h, and r as the picture shows.We state the Pythagorean theorem for this triangle and solve for h.

$$\begin{aligned} r^2+h^2 &= a^2 \\ 10^2+h^2 &= 26^2 \\ h^2+100 &= 676 \\ h^2 &= 576 \\ h &= \pm 24 \end{aligned}$$



The negative value is ruled out, and so the height is $24\,\mathrm{m}$.

9. An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.



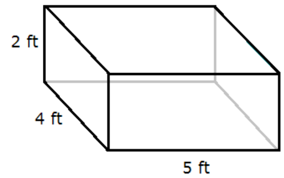
Solution: Consider the picture shown. Let r denote the radius of the semi-circle. We are asked to find the value of r. $AD = 1$ and $AB = 7$ were given.

Clearly $BC = r$, and this problem becomes easy once we realize that $AC = r - 1$ since the entire line segment CD is the radius r.Now we state the Pythagorean theorem on the right triangle ABC and solve for r.

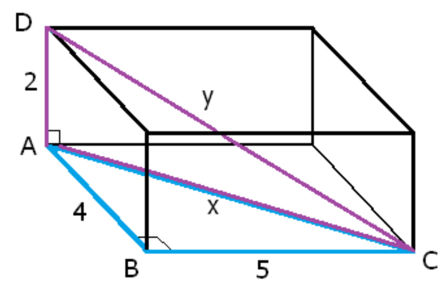
$$\begin{aligned} (r-1)^2 + 7^2 &= r^2 \\ r^2 - 2r + 1 + 49 &= r^2 \quad \text{subtract } r^2 \\ -2r + 50 &= 0 \quad \text{add } 2r \\ 50 &= 2r \quad \text{divide by 2} \\ 25 &= r \end{aligned}$$

Thus the maximum height of the arch is 25 feet.

10. Find the length of the longest line segment (called the main diagonal) in the rectangular prism shown on the picture.



Solution: We will apply the Pythagorean theorem twice. Let us label the points and sides we will use on the picture first.
We will find x using the Pythagorean theorem in triangle ABC.



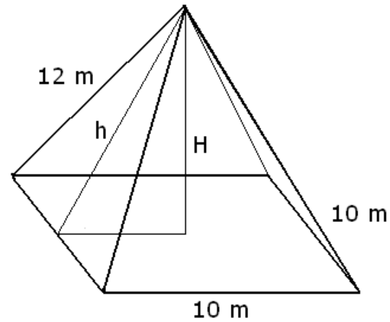
$$\begin{aligned} x^2 &= 4^2 + 5^2 \\ x^2 &= 41 \\ x &= \pm\sqrt{41} \qquad \Longrightarrow \qquad x = \sqrt{41} \end{aligned}$$

Now we can find y stating the Pythagorean theorem on triangle ACD.

$$\begin{aligned} 2^2 + x^2 &= y^2 \\ 2^2 + \left(\sqrt{41}\right)^2 &= y^2 \\ 45 &= y^2 \\ \pm\sqrt{45} &= y^2 \quad \Longrightarrow y = \sqrt{45} \end{aligned}$$

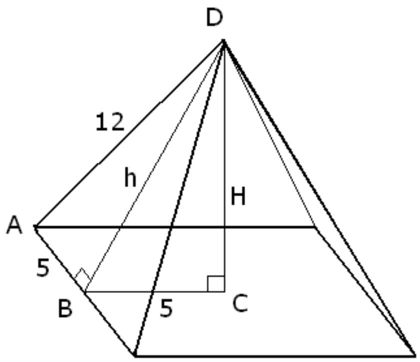
Note: Our result is actually $\sqrt{2^2 + 4^2 + 5^2}$. Indeed, we can see that the length of the main diagonal in a rectangular prism.with sides x, y, and z is $L = \sqrt{x^2 + y^2 + z^2}$. This is sometimes called the 3-dimensional Pythagorean theorem.

11. A pyramid (shown on the picture) has a square base with sides $10\,\mathrm{m}$ (meters) long. The other faces of the pyramid are isosceles triangles with sides $10\,\mathrm{m}$, $12\,\mathrm{m}$, and $12\,\mathrm{m}$.



a) Find the exact value of h, the length of the height in a triangular face.

Solution: Let us label some of the vertices as shown on the figure.



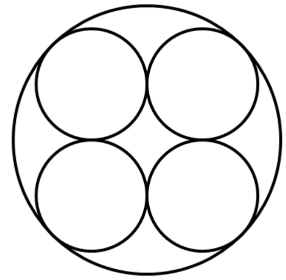
We can find the value of h by stating the Pythagorean theorem on the right triangle ABD. The hypotenuse is the side opposite the right angle.

$$\begin{aligned}(5\,\mathrm{m})^2 + h^2 &= (12\,\mathrm{m})^2\\ 25\,\mathrm{m}^2 + h^2 &= 144\,\mathrm{m}^2\\ h^2 &= 119\,\mathrm{m}^2\\ h &= \pm\sqrt{119\,\mathrm{m}^2} \quad\Longrightarrow\quad h = \sqrt{119}\,\mathrm{m}\end{aligned}$$

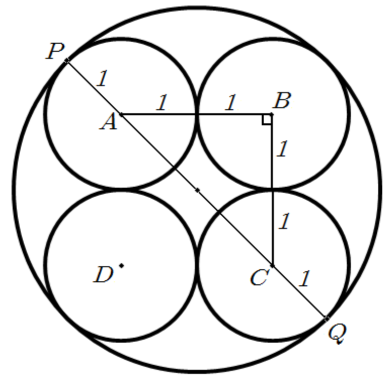
b) Find the exact value of H, the length of the height of the pyramid.
Solution: Let use the labels shown on the figure above. We can find the value of H by stating the Pythagorean theorem on the right triangle BCD. The hypotenuse is the side opposite the right angle.

$$\begin{aligned}(5\,\mathrm{m})^2 + H^2 &= h^2\\ (5\,\mathrm{m})^2 + H^2 &= \left(\sqrt{119}\,\mathrm{m}\right)^2\\ 25\,\mathrm{m}^2 + H^2 &= 119\,\mathrm{m}^2\\ H^2 &= 94\,\mathrm{m}^2\\ H &= \pm\sqrt{94\,\mathrm{m}^2} \quad\Longrightarrow\quad H = \sqrt{94}\,\mathrm{m}\end{aligned}$$

12. Four identical circles are arranged in a fifth circle as shown on the picture. Compute the exact value of the big circle if the smaller circles have radius 1.



Solution: Let us first draw a picture with the centers of all five circles indicated as shown on the picture. Please note that PQ means the line segment determined by points P and Q; and $\overline{PQ}$ means the length of that line segment.
Let R denote the radius of the larger circle and A, B, C, and D denote the centers of the small circles as shown. If we extend line segment AC to intersect the big circle, we obtain the points P and Q: Line segment PQ is the diagonal of the larger circle, and so $\overline{PQ} = 2R$. Line segment PQ is also the sum of three line segments:



$$PQ = PA + AC + CQ \quad \text{and so} \quad \overline{PQ} = 2R = \overline{PA} + \overline{AC} + \overline{CQ}$$

Line segments PA and CQ are 1 unit long because they are each a radius in a smaller circle. We will compute $\overline{AC}$ by the Pythagorean theorem, stated for triangle ABC. The length of ine segments AB and BC are both 2 units long because they measure the radius twice. So

$$\begin{aligned} 2^2 + 2^2 &= \left(\overline{AC}\right)^2 \\ 8 &= \left(\overline{AC}\right)^2 \\ \pm\sqrt{8} &= \overline{AC} \qquad \Longrightarrow \qquad \overline{AC} = \sqrt{8} = 2\sqrt{2} \end{aligned}$$

In this problem it is important that $\sqrt{8}$ can be simplified using the identity $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

We are now ready to compute R.

$$\begin{aligned} 2R &= \overline{PA} + \overline{AC} + \overline{CQ} \\ 2R &= 1 + 2\sqrt{2} + 1 \\ 2R &= 2 + 2\sqrt{2} \\ R &= \frac{2 + 2\sqrt{2}}{2} = \frac{2\left(1 + \sqrt{2}\right)}{2} = 1 + \sqrt{2} \end{aligned}$$

And so $\boxed{R = 1 + \sqrt{2}}$.

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