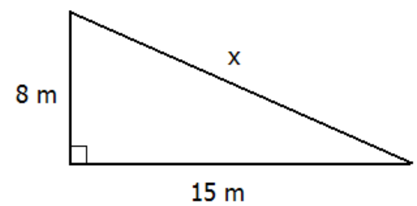
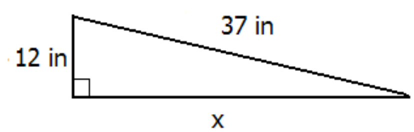
Sample Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.
   a) 6 cm, 10 cm, and 8 cm   b) 7 ft, 15 ft, and 50 ft   c) 4 m, 5 m, and 6 m

2. Find the hypotenuse of the triangle shown on the figure below.



3. Find the missing leg of the right triangle shown on the picture below.

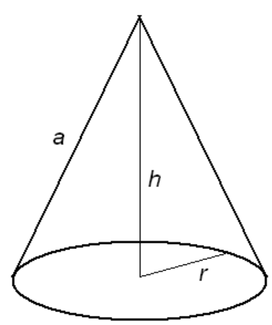


4. Find the distance between $(3,8)$ and $(8,-4)$.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

6. The hypotenuse of a right triangle is 68 cm. The difference between the other two sides is 28 cm. Find the sides of the triangle.

7. Find the height h of the cone shown on the picture below, if the base has a radius of 10 m and $a = 26$ m.

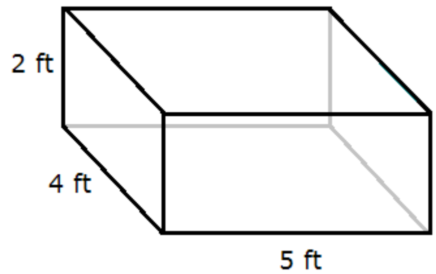


8. Find the height of an equilateral triangle with sides 10 m.

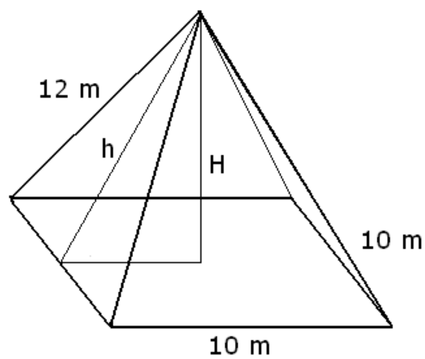
9. Suppose that C is the center of a circle with radius 15 feet. Let P be a point at a distance of 39 feet from C. From P, we draw a tangent line to the circle. Let Q be the point of tangency. Compute the distance between P and Q.

10. An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.

11. Find the length of the longest line segment (called the main diagonal) in the rectangular prism shown on the picture below.

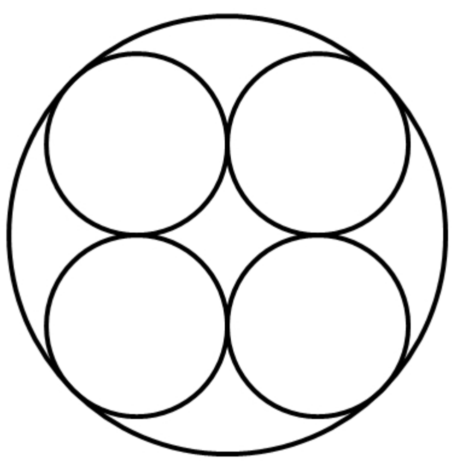


12. A pyramid (shown on the picture below) has a square base with sides 10 m (meters) long. The other faces of the pyramid are isosceles triangles with sides 10 m, 12 m, and 12 m.



a) Find the exact value of h, the length of the height in a triangular face.
b) Find the exact value of H, the length of the height of the pyramid.

13. Four identical circles are arranged in a fifth circle as shown on the picture below. Compute the exact value of the radius of the big circle if the smaller circles have radius 1.

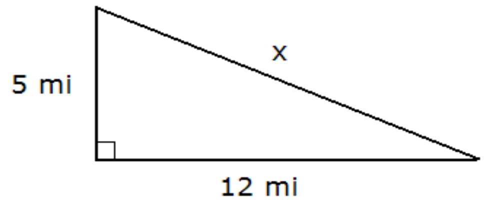


Practice Problems

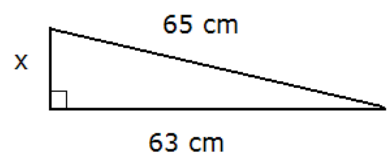
1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

   a) 2 cm, 7 cm, and 1 cm   b) 37 ft, 12 ft, and 35 ft   c) 6 m, 7 m, and 8 m

2. Find the hypotenuse of the triangle shown on the figure below.



3. Find the missing leg of the right triangle shown on the picture below.



4. Find the length of the diagonal in a rectangle with sides 20 ft and 21 ft long.

5. Find the length of the diagonal of a square with sides 1 unit long.

6. The sides of an isosceles triangle are 25 m, 25 m, and 14 m long. Find the length of the height drawn to the 14 m long side.

7. Two sides of a right triangle are 8 cm and 17 cm long. Find the length of the missing side.

8. Find the distance between the points

   a) $(-2,-3)$ and $(3,1)$.   b) $(-9,-3)$ and $(15,4)$.

9. One leg of a right triangle is 9 cm. The difference between the other two sides is 1 cm. Find the length of all sides.

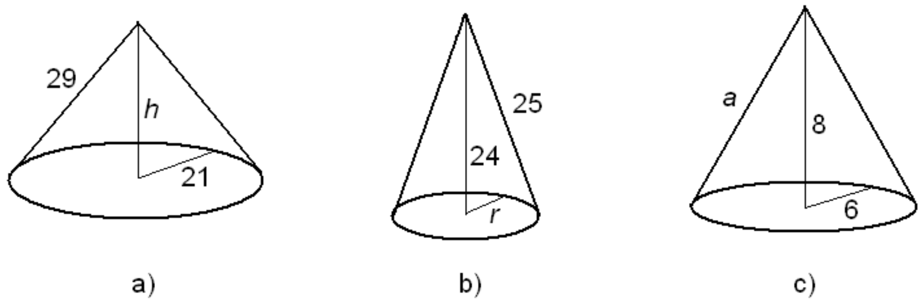
10. The hypotenuse of a right triangle is 50 in. The difference between the other two sides is 34 in. Find the length of all sides.

11. Consider a triangle with sides 7 unit, 7 unit, and 12 units long.
    a) Compute the exact value of the height drawn to the longest side.
    b) Compute the exact value of the area of the triangle.

12. Find the length of the main diagonal in a rectangular prism with sides

    a) 2 m, 10 m, and 11 m   b) 5 ft, 7 ft, and 1 ft

13. Find the missing lengths indicated on the picture below.



14. The height of a regular (same as equilateral) triangle is 6 feet. How long are its sides? Give the exact value of the answer.

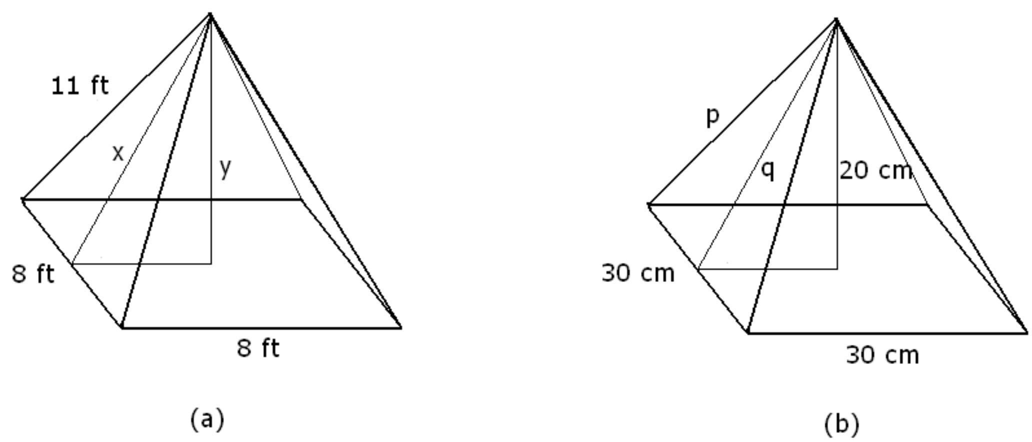
15. The diagonal of a square is 80 meters long. How long are the sides?

16. Suppose that C is a center of a circle and P is a point (outside of the circle) 9 units away from C. We draw a tangent line to the circle from point P. Let Q be the point of tangency. Given that line segment PQ is 8 units long, compute the exact value of
a) the radius of the circle. b) the area of triangle CPQ.

17. An arch is in the shape of a semicircle. At a point along the base 2 meters from an end of the arch, the height of the arch is 10 meters. Find the maximum height of the arch.

18. How long is the main diagonal in a cube of sides 1 meter long?

19. Find the exact value of the missing lengths, labeled
a) x and y b) p and q

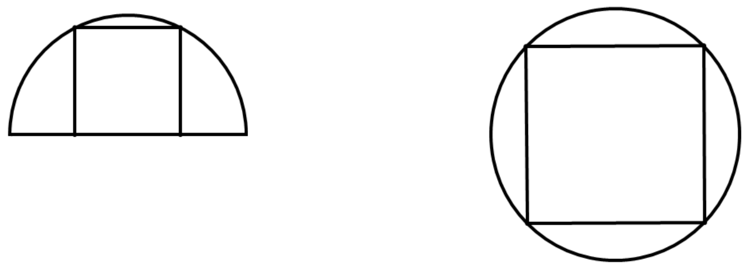


20. Consider a pyramid with a square base. The base has sides each 20 meters long. All other edges are 20 meters long.

a) Compute the exact value of the height of the pyramid.

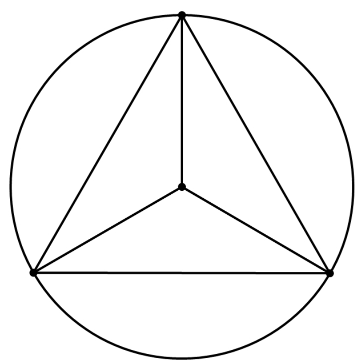
b) The volume of a pyramid can be computed as $V = \frac{1}{3}Bh$ where B is the area of its base and h is its height. Compute the volume of this pyramid. Give both the exact value and an approximation, accurate up to four or more digits after the decimal point.

21. a) Compute the area of a square written into a semicirle with radius 1.
 b) Compute the area of a square written into a cirle with radius 1.

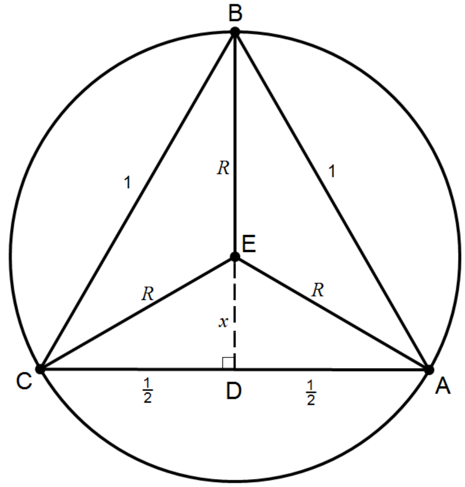


a) b)

22. Consider a regular triangle width sides 1 unit long, written into a circle as shown on the picture below.
 a) Find the exact value of the radius of the circle.
 b) The center of the circle splits the height of the traingle into two parts. What is the ratio of the shorter part to the longer part?



Hint: Consider the picture shown below. State the Pythagorean theorem on triangles BCD and CDE and solve the system.



Sample Problems - Answers

1. a) There is a right angle opposite the 10 cm long side. b) not even a triangle c) not a right triangle

2. 17 m 3. 35 inches 4. 13 units 5. 20 units 6. 32 cm and 60 cm 7. 24 m 8. $5\sqrt{3}$ m

9. 36 feet 10. 25 ft 11. $\sqrt{45}$ ft 12. $h=\sqrt{119}$ m $H=\sqrt{94}$ m 13. $1+\sqrt{2}$

Practice Problems - Answers

1. a) not even a triangle b) There is a right angle opposite the 37 ft long side. c) not a right triangle

2. 13 mi 3. 16 cm 4. 29 ft 5. $\sqrt{2}$ units 6. 24 m 7. 15 cm or $\sqrt{353}$ cm

8. a) $\sqrt{41}$ units b) 25 units 9. 9 cm, 40 cm, and 41 cm 10. 14 in, 48 in, and 50 in

11. a) $\sqrt{13}$ unit b) $6\sqrt{13}$ unit^2 12. a) 15 m b) $\sqrt{75}$ ft 13. a) 20 m b) 7 m c) 10 m

14. $4\sqrt{3}$ ft 15. $40\sqrt{2}$ m 16. a) $\sqrt{17}$ unit b) $4\sqrt{17}$ unit^2 17. 26 m 18. $\sqrt{3}$ m

19. a) $x=\sqrt{105}$ ft $y=\sqrt{89}$ ft b) $q=25$ cm, $p=\sqrt{850}$ cm $=5\sqrt{34}$ cm

20. a) $10\sqrt{2}$ m b) $\dfrac{4000}{3}\sqrt{2}\,\text{m}^3 \approx 1885.618\,08\,\text{m}^3$ 21. a) $\dfrac{4}{5}$ unit^2 b) 2 unit^2

22. a) $R=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$ b) 1 to 2

Sample Problems - Solutions

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

a) 6 cm, 10 cm, and 8 cm
Solution: The longest side is 10 cm long. Thus, only this side can be the hypotenuse. First we check the triangle-inequality: the two shorter sides should add up to a number greater than the logest side. $6+8=14$ and $14>10$, so this triangle does exist. Now we use the Pythagorean theorem to check for a right angle:

$$6^2+8^2 \stackrel{?}{=} 10^2$$

We get that the two quantities are equal, thus this triangle has a right angle. As always, it is opposite the longest side that, in this case, is 10 cm long.

b) 7 ft, 15 ft, and 50 ft
Solution: The longest side is 50 ft long. Thus, only this side can be the hypotenuse. First we check the triangle-inequality: the two shorter sides should add up to a number greater than the logest side. $7+15=22$ and $22 \not> 50$, so this triangle does not even exist, let alone has a right angle.

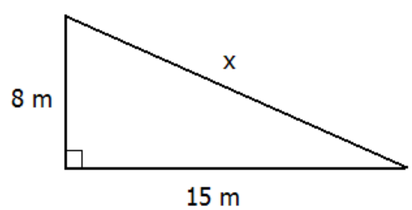
c) 4 m, 5 m, and 6 m
Solution: The longest side is 6 m long. Thus, only this side can be the hypotenuse. First we check the triangle-inequality: the two shorter sides should add up to a number greater than the logest side. $4+5=9$ and $9>6$, so this triangle does exist. Now we use the Pythagorean theorem to check for a right angle:

$$4^2+5^2 \stackrel{?}{=} 6^2$$

$$\begin{aligned} \text{LHS} &= 4^2+5^2=16+25=41 \\ \text{RHS} &= 6^2=36 \qquad\qquad \text{LHS} \neq \text{RHS} \end{aligned}$$

We get that the two quantities are not equal, thus this triangle does not have a right angle.

2. Find the hypotenuse of the triangle shown on the figure below.



Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned} 8^2+15^2 &= x^2 \\ 289 &= x^2 \\ \pm 17 &= x \end{aligned}$$

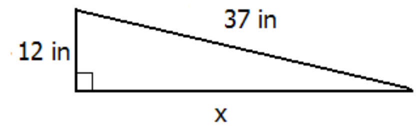
Since distance can not be negative, -17 is ruled out. The answer is 17 m.

Please note that the step taking us from $x^2=289$ to $x=\pm 17$ is a very nice shortcut. The tradiotional way of solving quadratic equations is to reduce one side to zero, factor, and apply the zero product rule.

$$\begin{aligned} x^2 &= 289 \\ x^2-289 &= 0 \\ x^2-17^2 &= 0 \\ (x+17)(x-17) &= 0 \qquad \Longrightarrow \qquad x=-17 \text{ or } x=17 \end{aligned}$$

Students are encouraged to use the shorter version, **as long as they don't make the serious algebraic error** of concluding from $x^2 = 289$ that $x = 17$. While in the context of the geometry the negative solution is not possible, the equation $x^2 = 289$ has two solutions, 17 and -17.

3. Find the missing leg of the right triangle shown on the picture below.

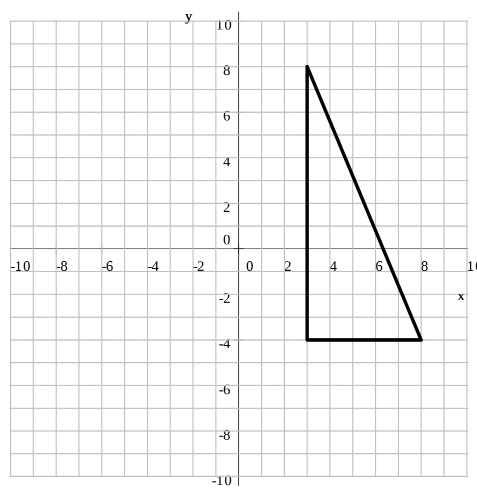


Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned}(12\,\text{in})^2 + x^2 &= (37\,\text{in})^2 \\ x^2 + 144\,\text{in}^2 &= 1369\,\text{in}^2 && \text{subtract } 144\,\text{in}^2 \\ x^2 &= 1225\,\text{in}^2 && \sqrt{1225} = 35 \\ x &= \pm 35\,\text{in}\end{aligned}$$

Since distance can not be negative, $-35\,\text{in}$ is ruled out. The answer is 35 inches.

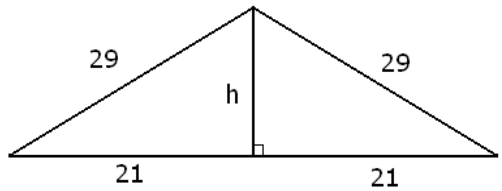
4. Find the distance between $(3, 8)$ and $(8, -4)$.
Solution: We graph the points, they determine a right triangle as shown below. We can compute the distance as the hypotenuse of the right triangle. How long are the legs?
Algebra: $8 - 3 = 5$ and $8 - (-4) = 12$.
The difference will alwys work. Even if we get -5 instead of 5, it will not matter since we will square it in the Pythagorean theorem.
Geometry: From 3 to 8 we have to step 5 units up. From -4 to 8 : first we step 4 to get from -4 to 0. Then another 8 steps to 8, and so $4 + 8 = 12$ steps. The message here is that the algebra and geometry will always agree.
The legs are 5 and 12 units long, and we need to find the hypotenuse.



$$\begin{aligned}5^2 + 12^2 &= x^2 \\ 25 + 144 &= x^2 \\ 169 &= x^2 \\ \pm 13 &= x\end{aligned}$$

Since distances are never negative, -13 is ruled out and so the answer is 13 units.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.
Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles as shown on the picture below.

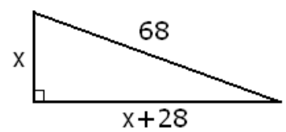


The height now can be easily computed via the Pythagorean theorem.

$$\begin{aligned} 21^2 + h^2 &= 29^2 \\ 441 + h^2 &= 841 \\ h^2 &= 400 \\ h &= \pm 20 \quad \Longrightarrow \quad h = 20 \end{aligned}$$

Again, the negative solution of the equation is ruled out because distances can not be negative. The height belonging to the base is 20 units long.

6. The hypotenuse of a right triangle is 68 cm. The difference between the other two sides is 28 cm. Find the sides of the triangle.
Solution: Let x denote the shorter leg. Then the other leg is $x + 28$ cm long.



We state the Pythagorean theorem for the triangle, and solve the quadratic equation for x.

$$\begin{aligned} x^2 + (x+28)^2 &= 68^2 && \text{FOIL out } (x+28)^2 \\ x^2 + x^2 + 56x + 784 &= 4624 && \text{combine like terms} \\ 2x^2 + 56x + 784 &= 4624 && \text{subtract } 4624 \\ 2x^2 + 56x - 3840 &= 0 && \text{factor out } 2 \\ 2\left(x^2 + 28x - 1920\right) &= 0 && \text{divide by } 2 \\ x^2 + 28x - 1920 &= 0 \end{aligned}$$

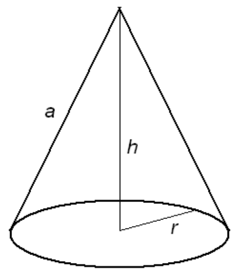
We factor by completing the square. Since half of the linear coefficient is 14, we will work with $(x+14)^2 = x^2 + 28x + 196$

$$\begin{aligned} \underbrace{x^2 + 28x + 196} - 196 - 1920 &= 0 \\ (x+14)^2 - 2116 &= 0 \\ (x+14)^2 - 46^2 &= 0 \\ (x+14+46)(x+14-46) &= 0 \\ (x+60)(x-32) &= 0 \quad \Longrightarrow \quad x_1 = -60 \quad \text{and} \quad x_2 = 32 \end{aligned}$$

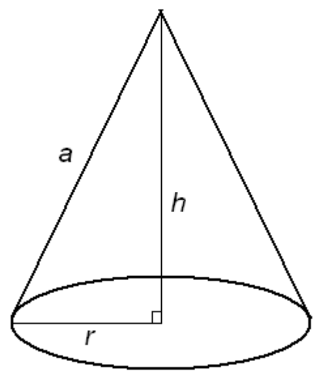
Since distances are never negative, -60 is ruled out. If the shortest side is 32 cm, the other side is $32\,\text{cm} + 28\,\text{cm} = 60\,\text{cm}$. Thus the solution is 32 cm and 60 cm. We check:

$$60 - 32 = 28 \checkmark \quad \text{and} \quad 60^2 + 32^2 = 3600 + 1024 = 4624 = 68^2 \checkmark$$

7. Find the height h of the cone shown on the picture below, if the base has a radius of $10\,\mathrm{m}$ and $a = 26\,\mathrm{m}$.



Solution: There is a right triangle formed by a, h, and r as the picture below shows.

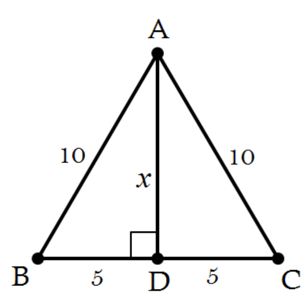


We state the Pythagorean theorem for this triangle and solve for h.

$$\begin{aligned}
r^2 + h^2 &= a^2 \\
h^2 &= a^2 - r^2 \\
h &= \pm\sqrt{a^2 - r^2} && \text{negative value is ruled out} \\
h &= \sqrt{a^2 - r^2} && r = 10\,\mathrm{m} \text{ and } a = 26\,\mathrm{m} \\
h &= \sqrt{(26\,\mathrm{m})^2 - (10\,\mathrm{m})^2} = \sqrt{676\,\mathrm{m}^2 - 100\,\mathrm{m}^2} = \sqrt{576\,\mathrm{m}^2} = 24\,\mathrm{m}
\end{aligned}$$

Thus the height is $24\,\mathrm{m}$.

8. Find the height of an equilateral triangle with sides $10\,\mathrm{m}$.
Solution: Consider the picture shown below.



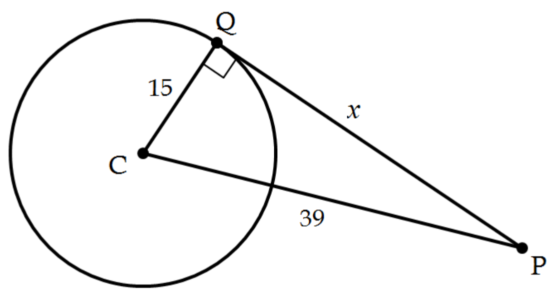
Since the triangle is equilateral, all three sides are 10 meters long. When we draw in the height belonging to side BC, it splits BC in a half, thus $BD = 5$. We state the Pythagorean theorem for the right triangle ABD

and solve for x.

$$\begin{aligned}(5\,\mathrm{m})^2 + x^2 &= (10\,\mathrm{m})^2\\ 25\,\mathrm{m}^2 + x^2 &= 100\,\mathrm{m}^2\\ x^2 &= 75\,\mathrm{m}^2\\ x &= \pm\sqrt{75}\,\mathrm{m}\end{aligned}$$

Since distances can negative, $-\sqrt{75}\,\mathrm{m}$ is ruled out. Please note that $\sqrt{75}$ can be simplified as $5\sqrt{3}$ The solution, $\sqrt{75}\,\mathrm{m}$ or $5\sqrt{3}\,\mathrm{m}$ is acceptable in both forms.

9. Suppose that C is the center of a circle with radius 15 feet. Let P be a point at a distance of 39 feet from C. From P, we draw a tangent line to the circle. Let Q be the point of tangency. Compute the distance between P and Q.
Solution: To solve this problem, we need to know that **in case of circles, the radius drawn to the point of tangency is perpendicular to the tangent line.** Consider the picture shown below. We denote the length of line segment PQ by x.

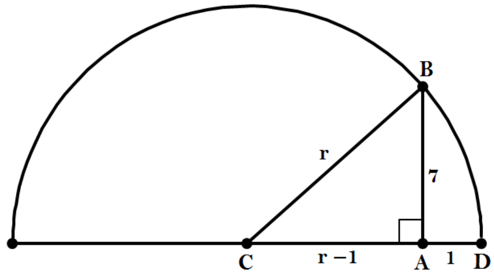


We will state the Pythagorean theorem for the right triangle PQC and solve for x.

$$\begin{aligned}15^2 + x^2 &= 39^2\\ 225 + x^2 &= 1521 \qquad \text{subtract } 225\\ x^2 &= 1296\\ x &= \pm 36\end{aligned}$$

Since we are looking for a distance, the negative solution is ruled out. Thus the answer is 36 feet.

10. An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.
Solution: Consider the picture below.

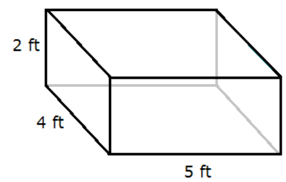


Let r denote the radius of the semi-circle. We are asked to find the value of r. $AD = 1$ and $AB = 7$ were given. Clearly $BC = r$, and this problem becomes easy once we realize that $AC = r - 1$ since the entire line segment CD is the radius r. Now we state the Pythagorean theorem on the right triangle ABC and solve for r.

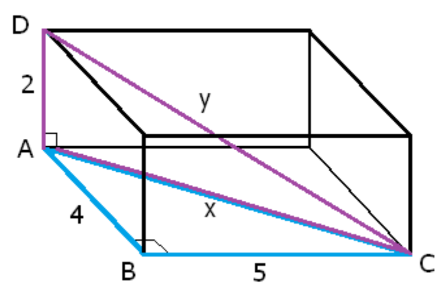
$$\begin{aligned}(r-1)^2 + 7^2 &= r^2 \\ r^2 - 2r + 1 + 49 &= r^2 \qquad \text{subtract } r^2 \\ -2r + 50 &= 0 \qquad \text{add } 2r \\ 50 &= 2r \qquad \text{divide by 2} \\ 25 &= r\end{aligned}$$

Thus the maximum height of the arch is 25 feet.

11. Find the length of the longest line segment (called the main diagonal) in the rectangular prism shown on the picture below.



Solution: We will apply the Pythagorean theorem twice. Let us label the points and sides we will use on the picture first.



We will find x using the Pythagorean theorem in triangle ABC.

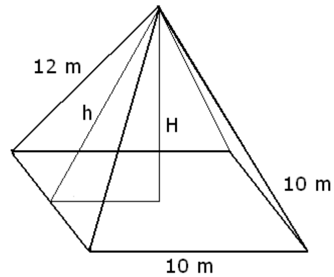
$$\begin{aligned}x^2 &= 4^2 + 5^2 \\ x^2 &= 41 \\ x &= \pm\sqrt{41} \qquad \Longrightarrow \qquad x = \sqrt{41}\end{aligned}$$

Now we can find y stating the Pythagorean theorem on triangle ACD.

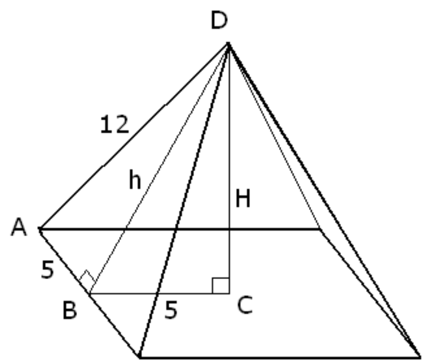
$$\begin{aligned}2^2 + x^2 &= y^2 \\ 2^2 + \left(\sqrt{41}\right)^2 &= y^2 \\ 45 &= y^2 \\ \pm\sqrt{45} &= y^2 \quad \Longrightarrow y = \sqrt{45}\end{aligned}$$

Note: Our result is actually $\sqrt{2^2 + 4^2 + 5^2}$. Indeed, we can see that the length of the main diagonal in a rectangular prism.with sides x, y, and z is $L = \sqrt{x^2 + y^2 + z^2}$. This is sometimes called the 3-dimensional Pythagorean theorem.

12. A pyramid (shown on the picture below) has a square base with sides 10 m (meters) long. The other faces of the pyramid are isosceles triangles with sides 10 m, 12 m, and 12 m.



a) Find the exact value of h, the length of the height in a triangular face.
Solution: Let us label some of the vertices as shown on the figure below.



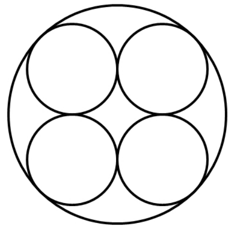
We can find the value of h by stating the Pythagorean theorem on the right triangle ABD. The hypotenuse is the side opposite the right angle.

$$\begin{aligned}(5\,\mathrm{m})^2+h^2 &= (12\,\mathrm{m})^2\\ 25\,\mathrm{m}^2+h^2 &= 144\,\mathrm{m}^2\\ h^2 &= 119\,\mathrm{m}^2\\ h &= \pm\sqrt{119\,\mathrm{m}^2} \quad\Longrightarrow\quad h=\sqrt{119}\,\mathrm{m}\end{aligned}$$

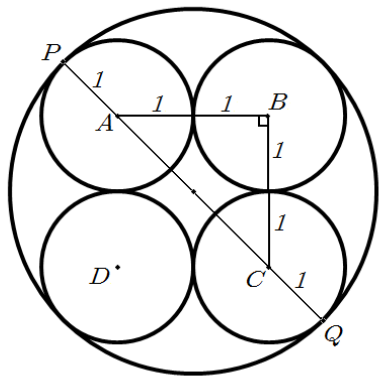
b) Find the exact value of H, the length of the height of the pyramid.
Solution: Let use the labels shown on the figure above. We can find the value of H by stating the Pythagorean theorem on the right triangle BCD. The hypotenuse is the side opposite the right angle.

$$\begin{aligned}(5\,\mathrm{m})^2+H^2 &= h^2\\ (5\,\mathrm{m})^2+H^2 &= \left(\sqrt{119}\,\mathrm{m}\right)^2\\ 25\,\mathrm{m}^2+H^2 &= 119\,\mathrm{m}^2\\ H^2 &= 94\,\mathrm{m}^2\\ H &= \pm\sqrt{94\,\mathrm{m}^2} \quad\Longrightarrow\quad H=\sqrt{94}\,\mathrm{m}\end{aligned}$$

13. Four identical circles are arranged in a fifth circle as shown on the picture below. Compute the exact value of the big circle if the smaller circles have radius 1.



Solution: Let us first draw a picture with the centers of all five circles indicated as shown on the picture



(Please note that PQ means the line segment determined by points P and Q; and $\overline{PQ}$ means the length of that line segment.)

Let R denote the radius of the larger circle and A, B, C, and D denote the centers of the small circles as shown. If we extend line segment AC to intersect the big circle, we obtain the points P and Q: Line segment PQ is the diagonal of the larger circle, and so $\overline{PQ} = 2R$. Line segment PQ is also the sum of three line segments:

$$PQ = PA + AC + CQ \quad \text{and so} \quad \overline{PQ} = 2R = \overline{PA} + \overline{AC} + \overline{CQ}$$

Line segments PA and CQ are 1 unit long because they are each a radius in a smaller circle. We will compute $\overline{AC}$ by the Pythagorean theorem, stated for triangle ABC. The length of ine segments AB and BC are both 2 units long because they measure the radius twice. So

$$\begin{aligned} 2^2 + 2^2 &= \left(\overline{AC}\right)^2 \\ 8 &= \left(\overline{AC}\right)^2 \\ \pm\sqrt{8} &= \overline{AC} \qquad \Longrightarrow \qquad \overline{AC} = \sqrt{8} = 2\sqrt{2} \end{aligned}$$

In this problem it is important that $\sqrt{8}$ can be simplified using the identity $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

We are now ready to compute R.

$$\begin{aligned} 2R &= \overline{PA} + \overline{AC} + \overline{CQ} \\ 2R &= 1 + 2\sqrt{2} + 1 \\ 2R &= 2 + 2\sqrt{2} \qquad \Longrightarrow \qquad R = \frac{2 + 2\sqrt{2}}{2} = \frac{2\left(1 + \sqrt{2}\right)}{2} = 1 + \sqrt{2} \end{aligned}$$

And so $R = 1 + \sqrt{2}$.

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