Sample Problems

1. Consider the function $f(x) = \dfrac{(x+2)^2 x^5 (x-2)}{(x+4)\, x\, (x-2)^2}$

   a) Determine the discontinuities of f.

   b) State which of the discontinuities are holes and which are vertical asymptotes.

   c) Determine the end-behavior of f.

2. Redo problem 1) with the function $f(x) = \dfrac{(x+3)(x+1)\, x^2 (x-2)(x-5)^3}{(x+3)^4 (x+1)\, x\, (x-2)^2 (x-5)^2 (x-7)}$

3. Graph $f(x) = \dfrac{3x-5}{x+2}$

4. Graph $f(x) = \dfrac{-(x+2)(x+1)^2 (x-2)}{(x+1)\, x^2 (x-1)(x-2)}$

Practice Problems

Plot the graph each of the following functions.

1. a) $f(x) = \dfrac{-2x+1}{x-3}$ c) $f(x) = \dfrac{x-3}{-x+1}$

   b) $f(x) = \dfrac{5x-7}{3x+1}$ d) $f(x) = \dfrac{4x-1}{x-2}$

2. $f(x) = \dfrac{2(x-3)}{x-3}$

3. $f(x) = \dfrac{(x+4)(x+1)(x-3)}{(x+1)}$

4. $f(x) = \dfrac{(x-3)(x+1)}{(x+4)(x+1)}$

5. $f(x) = \dfrac{2(x+4)(x-3)}{(x+1)^2}$

6. $f(x) = \dfrac{1}{(x+4)(x+1)^2 (x-3)}$

7. $f(x) = \dfrac{-2(x+2)\, x\, (x-2)^3 (x-3)^2}{(x+1)^2 x^2 (x-2)^2 (x-3)^2}$

8. $f(x) = \dfrac{x(x+2)(x-3)}{(x-3)(x^2-1)}$

9. $f(x) = \dfrac{-(x+2)^2 (x+1)(x-2)^2 (x-4)(x-5)}{(x+1)\, x\, (x-1)^2 (x-2)(x-4)}$

10. $f(x) = \dfrac{(x+2)^3 x (x-1)(x-3)}{(x+3)(x+2)\, x\, (x-2)^4 (x-3)}$

11. $f(x) = \dfrac{(x+3)(x+2)(x+1)^2 x (x-1)^5}{(x+2)^5 (x+1)\, x\, (x-1)(x-2)^3}$

12. $f(x) = \dfrac{(x+5)^3 (x+3)\, x\, (x-2)^4 (x-4)^4}{(x+5)(x+3)^4 x (x-2)^3 (x-4)^6}$

Sample Problems - Answers

1. a) at $x = -4, 0$, and 2

   b) f has a hole at $x = 0$ and a vertical asymptote at $x = -4$ and $x = 2$

   c) $\lim\limits_{x \to -\infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$.

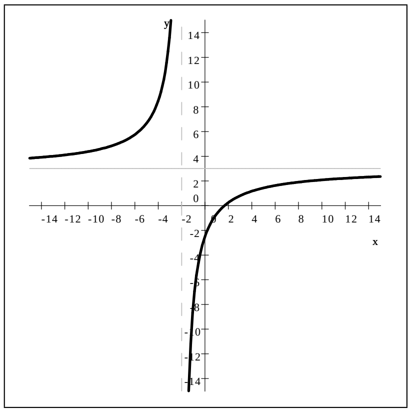
2. a) at $x = -3, -1, 0, 2, 5$, and 7

   b) f has holes at $x = -1, 0, 5$ and vertical asymptotes at $x = -3, 2, 7$

   c) $\lim\limits_{x \to -\infty} f(x) = 0^-$ and $\lim\limits_{x \to \infty} f(x) = 0^+$.

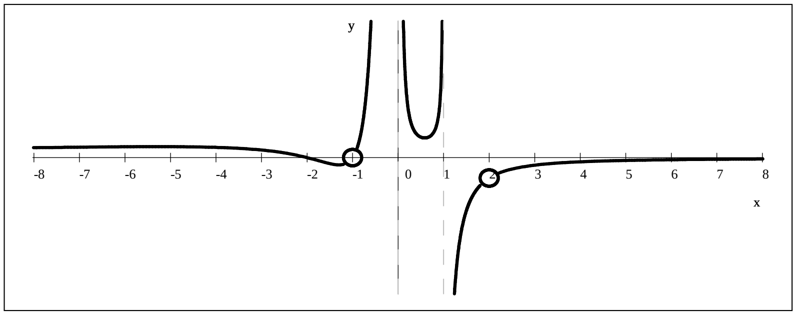
3. Graph $f(x) = \dfrac{3x-5}{x+2}$

   horizontal asymptote: $y = 3$; vertical asymptote: $x = -2$; intercepts: $\left(0, -\dfrac{5}{2}\right)$ and $\left(\dfrac{5}{3}, 0\right)$



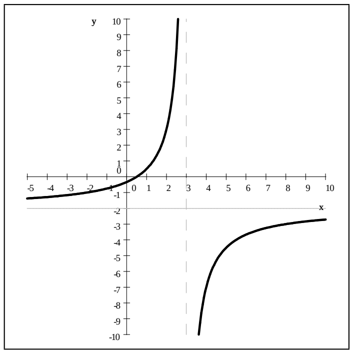
4. $f(x) = \dfrac{-(x+2)(x+1)^2(x-2)}{(x+1)x^2(x-1)(x-2)} = \begin{cases} \dfrac{-(x+2)(x+1)}{x^2(x-1)} & \text{if } x \neq -1, 2 \\ \text{undefined} & \text{if } x = -1, 2 \end{cases}$

   horizontal asymptote: $y = 0$; vertical asymptotes: $x = 0, 1$; intercepts: $(-2, 0)$

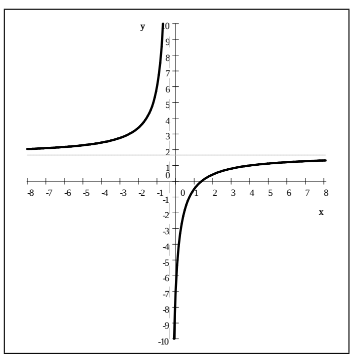


Practice Problems - Answers

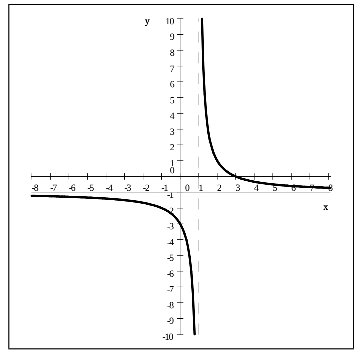
1. a) $f(x) = \dfrac{-2x+1}{x-3}$
horizontal asymptote: $y = -2$,
vertical asymptote: $x = 3$;
intercepts: $\left(0, -\dfrac{1}{3}\right)$ and $\left(\dfrac{1}{2}, 0\right)$



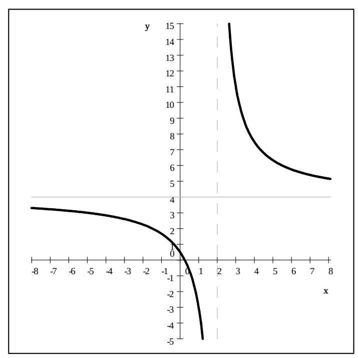
b) $f(x) = \dfrac{5x-7}{3x+1}$
horizontal asymptote: $y = \dfrac{5}{3}$,
vertical asymptote: $x = -\dfrac{1}{3}$;
intercepts: $(0, -7)$ and $\left(\dfrac{7}{5}, 0\right)$



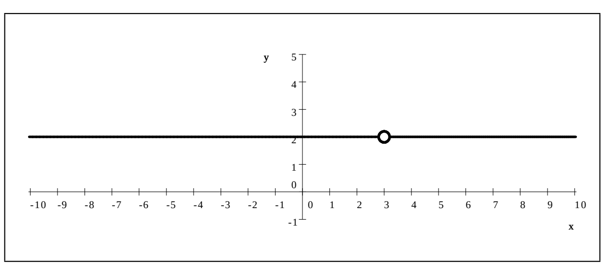
c) $f(x) = \dfrac{x-3}{-x+1}$
horizontal asymptote: $y = -1$,
vertical asymptote: $x = 1$;
intercepts: $(0, -3)$ and $(3, 0)$



d) $f(x) = \dfrac{4x-1}{x-2}$
horizontal asymptote: $y = 4$,
vertical asymptote: $x = 2$;
intercepts: $\left(0, \dfrac{1}{2}\right)$ and $\left(\dfrac{1}{4}, 0\right)$

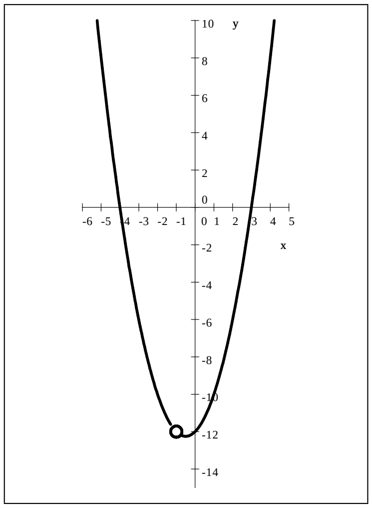


2. $f(x) = \dfrac{2(x-3)}{x-3} = \begin{cases} 2 & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases}$
hole at $x = 3$; intercepts: $(0, 3)$



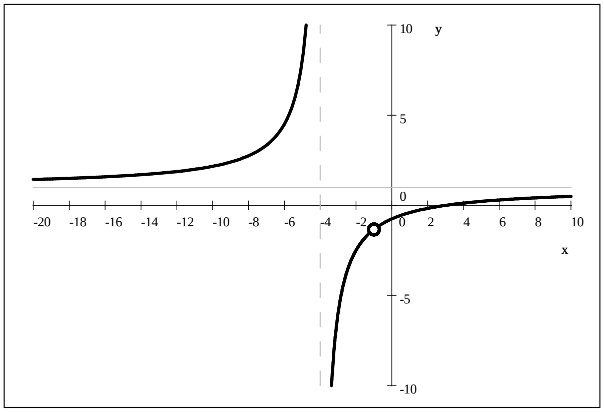
3. $f(x) = \dfrac{(x+4)(x+1)(x-3)}{(x+1)} = \begin{cases} (x+4)(x-3) & \text{if } x \neq -1 \\ \text{undefined} & \text{if } x = -1 \end{cases}$

hole: at $x = -1$; intercepts: $(0,-12)$, $(-4,0)$, and $(3,0)$



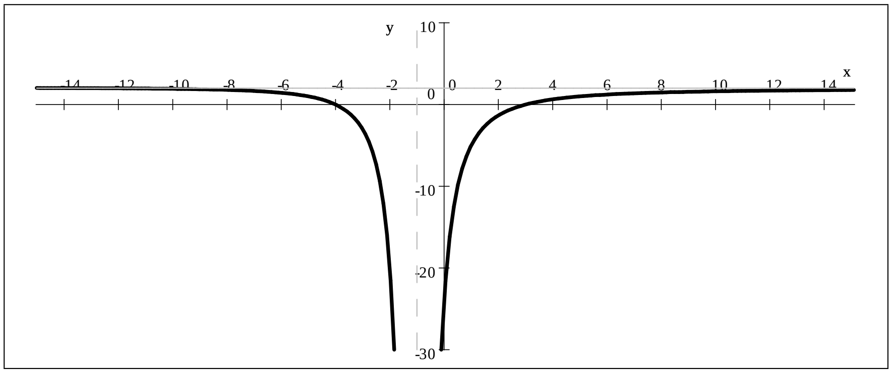
4. $f(x) = \dfrac{(x-3)(x+1)}{(x+4)(x+1)} = \begin{cases} \dfrac{x-3}{x+4} & \text{if } x \neq -1 \\ \text{undefined} & \text{if } x = -1 \end{cases}$

horizontal asymptote: $y = 1$; vertical asymptote: $x = -4$; hole: at $x = -1$; intercepts: $\left(0, -\dfrac{3}{4}\right)$, $(3,0)$



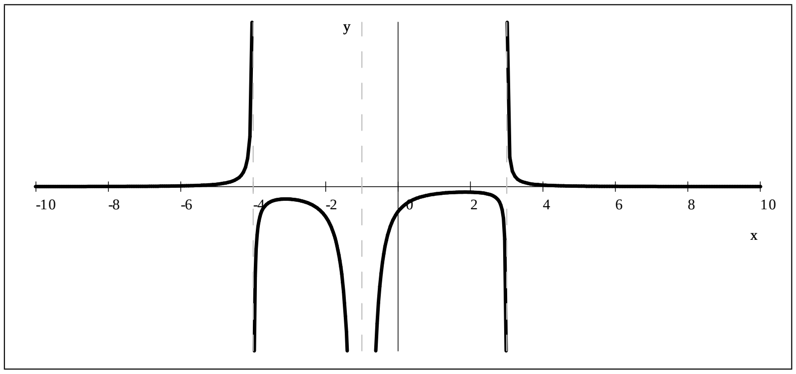
5. $f(x) = \dfrac{2(x+4)(x-3)}{(x+1)^2}$

horizontal asymptote: $y = 2$; vertical asymptote: $x = -1$; intercepts: $(-4,0)$ and $(3,0)$



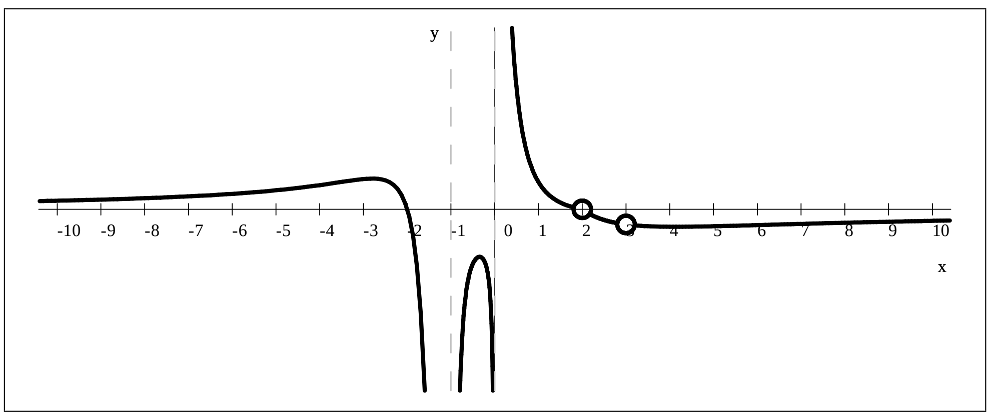
6. $f(x) = \dfrac{1}{(x+4)(x+1)^2(x-3)}$

horizontal asymptote: $y = 0$; vertical asymptotes: $x = -4, -1, 3$; intercepts: $\left(0, -\dfrac{1}{12}\right)$



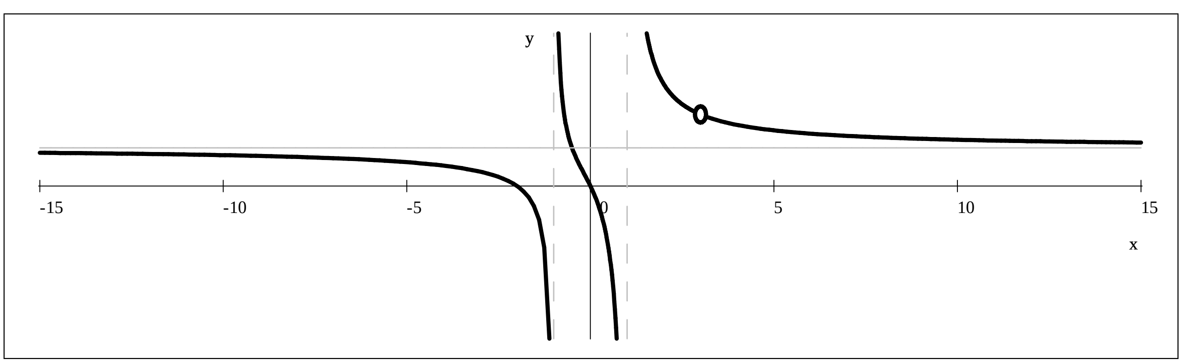
7. $f(x) = \dfrac{-2(x+2)x(x-2)^3(x-3)^2}{(x+1)^2 x^2 (x-2)^2 (x-3)^2} = \begin{cases} \dfrac{-2(x+2)(x-2)}{(x+1)^2 x} & \text{if } x \neq 2, 3 \\ \text{undefined} & \text{if } x = 2, 3 \end{cases}$

horizontal asymptote: $y = 0$; vertical asymptotes: $x = -1, 0$; holes at $x = 2, 3$, intercepts: $(-2, 0)$



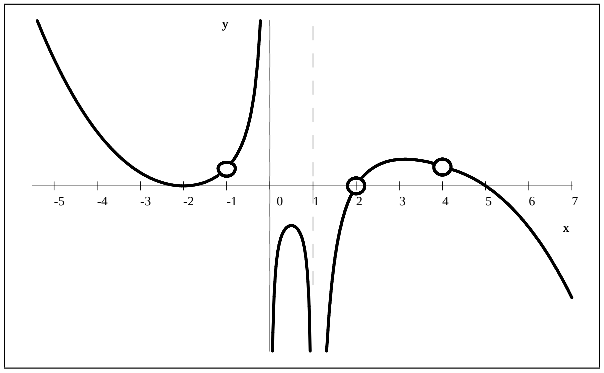
8. $f(x) = \dfrac{x(x+2)(x-3)}{(x-3)(x^2-1)} = \begin{cases} \dfrac{x(x+2)}{(x+1)(x-1)} & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases}$

horizontal asymptote: $y = 2$; vertical asymptotes: $x = -1, 1$; intercepts: $(-2, 0)$ and $(0, 0)$



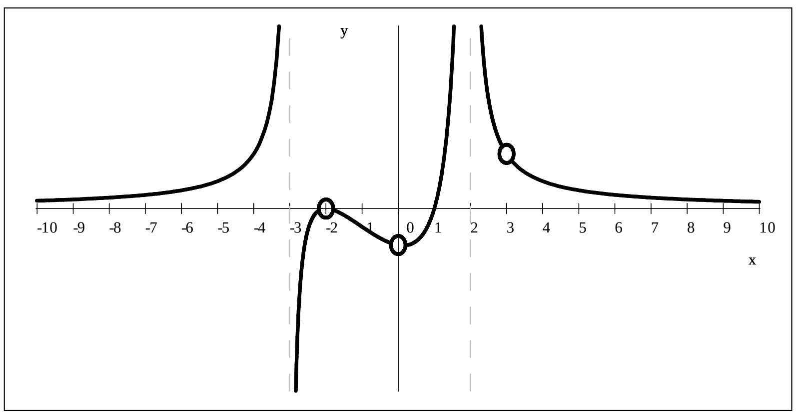
9. $f(x) = \dfrac{-(x+2)^2(x+1)(x-2)^2(x-4)(x-5)}{(x+1)x(x-1)^2(x-2)(x-4)} = \begin{cases} \dfrac{-(x+2)^2(x-2)(x-5)}{x(x-1)^2} & \text{if } x \neq -1, 2, 4 \\ \text{undefined} & \text{if } x = -1, 2, 4 \end{cases}$

vertical asymptotes: $x = 0, 1$; holes: at $x = -1, 2, 4$ intercepts: $(-2, 0)$, $(5, 0)$



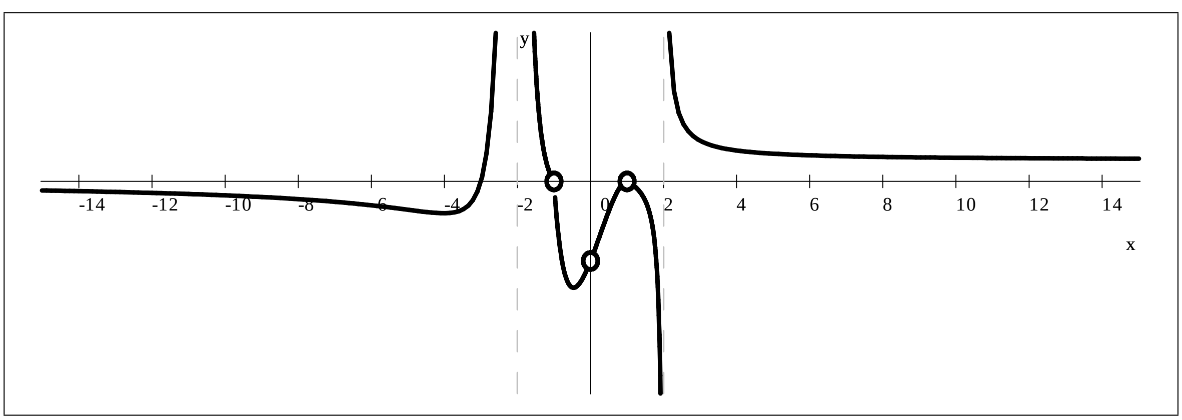
10. $f(x) = \dfrac{(x+2)^3 x(x-1)(x-3)}{(x+3)(x+2)x(x-2)^4(x-3)} = \begin{cases} \dfrac{(x+2)^2(x-1)}{(x+3)(x-2)^4} & \text{if } x \neq -2, 0, 3 \\ \text{undefined} & \text{if } x = -2, 0, 3 \end{cases}$

horizontal asymptote: $y = 0$; vertical asymptotes: $x = -3, 2$; holes at $x = -2, 0, 3$, intercepts: $(1, 0)$



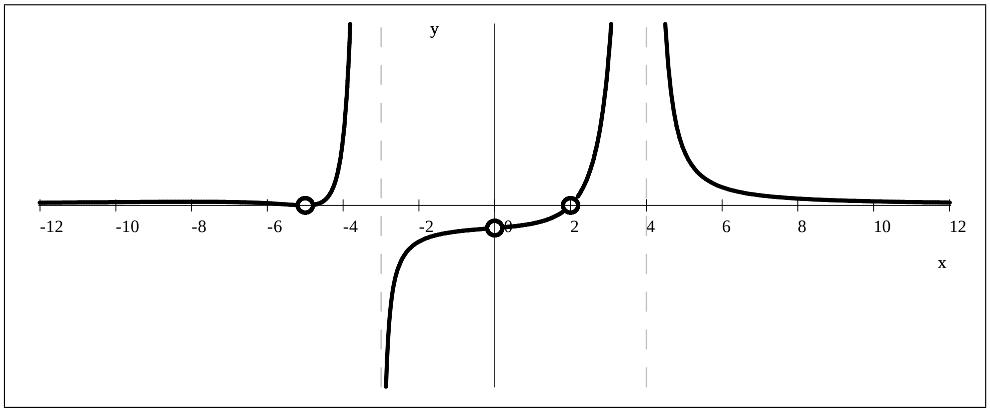
11. $f(x) = \dfrac{(x+3)(x+2)(x+1)^2 x(x-1)^5}{(x+2)^5(x+1)x(x-1)(x-2)^3} = \begin{cases} \dfrac{(x+3)(x+1)(x-1)^4}{(x+2)^4(x-2)^3} & \text{if } x \neq -1, 0, 1 \\ \text{undefined} & \text{if } x = -1, 0, 1 \end{cases}$

horizontal asymptote: $y = 0$; vertical asymptotes: $x = -2, 2$; holes at $x = -1, 0, 1$, intercepts: $(-3, 0)$



12. $f(x) = \dfrac{(x+5)^3(x+3)x(x-2)^4(x-4)^4}{(x+5)(x+3)^4 x(x-2)^3(x-4)^6} = \begin{cases} \dfrac{(x+5)^2(x-2)}{(x+3)^3(x-4)^2} & \text{if } x \neq -5, 0, 2 \\ \text{undefined} & \text{if } x = -1, 2 \end{cases}$

horizontal asymptote: $y = 0$; vertical asymptotes: $x = -3, 4$; holes at $x = -5, 0, 2$, no intercepts



Sample Problems - Solutions

1. Consider the function $f(x) = \dfrac{(x+2)^2 x^5 (x-2)}{(x+4)x(x-2)^2}$

a) Determine the discontinuities of f.

Solution: The discontinuities of rational functions are the zeroes of its denominator; in this case, they are at $x = -4, 0$, and 2.

b) State which of the discontinuities are holes and which are vertical asymptotes.

Solution: We first carefully simplify the function's equation. If we simplify as we always did before, we get that

$$f(x) = \frac{(x+2)^2 x^5 (x-2)}{(x+4)x(x-2)^2} = \frac{(x+2)^2 x^4}{(x+4)(x-2)}$$

This is incorrect. Recall that two functions are equal if they have the same assignment and the **same domain**. The two functions above have different domains: $x = 0$ is in the domain of the right-hand side but not in the left-hand side. Here is how we simplify carefully:

$$f(x) = \frac{(x+2)^2 x^5 (x-2)}{(x+4)x(x-2)^2} = \begin{cases} \dfrac{(x+2)^2 x^4}{(x+4)(x-2)} & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

Now the the two functions are truly equal since they have the same domain. We are ready to answer the question.

The zeroes of the denominator in the original function but not in the simplified one are the **holes**. The zeroes of the denominator of the simplified form is where the graph will have a **vertical asymptote**. So f as a hole at $x = 0$ and a vertical asymptote at $x = -4$ and $x = 2$.

c) Determine the end-behavior of f.

Solution: Recall that in case of polynomials, the end-behavior is completely determined by their leading terms. In case of rational functions, the end-behavior is entirely determined by the leading terms of the numerator and denominator. It does not matter whether we use the original or the simplified form of the function, so we will use the simplified form In the case of $f(x) = \dfrac{(x+2)^2 x^4}{(x+4)(x-2)}$, the leading term of the numerator is x^6 and that of the denominator is x^2. Thus, the end-behavior of f is the same as the end-behavior of $\dfrac{x^6}{x^2} = x^4$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^6}{x^2} = \lim_{x\to-\infty} x^4 = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^6}{x^2} = \lim_{x\to\infty} x^4 = \infty$$

2. Consider the function $f(x) = \dfrac{(x+3)(x+1)x^2(x-2)(x-5)^3}{(x+3)^4(x+1)x(x-2)^2(x-5)^2(x-7)}$

a) Determine the discontinuities of f.

Solution: The discontinuities of rational functions are the zeroes of its denominator: in case of f, they are at $x = -3, -1, 0, 2, 5$, and 7.

b) State which of the discontinuities are holes and which are vertical asymptotes.

Solution: We first carefully simplify the function's equation. Recall that two functions are equal if they have the same assignment and the **same domain**.

$$f(x) = \frac{(x+3)(x+1)x^2(x-2)(x-5)^3}{(x+3)^4(x+1)x(x-2)^2(x-5)^2(x-7)} = \begin{cases} \dfrac{x(x-5)}{(x+3)^3(x-2)(x-7)} & \text{if} \quad x \neq -1, 0, 5 \\[2ex] \text{undefined} & \text{if} \quad x = -1, 0, 5 \end{cases}$$

Now the two functions are truly equal. We are ready to answer the question.

The zeroes of the denominator in the original function but not in the simplified one are the **holes**. The zeroes of the denominator of the simplified form is where the graph will have a **vertical asymptote**. So the function f has a hole at $x = -1, 0$, and 5 and a vertical asymptote at $x = -3, 2$ and 7.

c) Determine the end-behavior of f.

Solution: The end-behavior is entirely determined by the leading terms of the numerator and denominator. We will use the simplified form to determine that In the case of $f(x) = \dfrac{x(x-5)}{(x+3)^3(x-2)(x-7)}$, the leading term of the numerator is x^2 and that of the denominator is x^5. Thus, the end-behavior of f is the same as the end-behavior of $\dfrac{x^2}{x^5} = \dfrac{1}{x^3}$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^5} = \lim_{x\to-\infty} \frac{1}{x^3} = 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^5} = \lim_{x\to\infty} \frac{1}{x^3} = 0$$

Please note that in this case, we will need more information to graph the function. In order to correctly graph this function, we also need to know the sign of $\dfrac{1}{x^3}$ for large positive and negative values of x. If x is negative, then so is $\dfrac{1}{x^3}$, and if x is positive, then so is $\dfrac{1}{x^3}$. So, to be precise, the end-behavior of f is

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^3} = 0^- \quad \text{and} \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^3} = 0^+$$

3. Graph $f(x) = \dfrac{3x-5}{x+2}$

Solution 1. The end-behavior can be determined by the quotient of the leading terms. Here is the actual computation:

$$\lim_{x\to-\infty} \frac{3x-5}{x+2} = \lim_{x\to-\infty} \frac{(3x)\left(1-\dfrac{5}{3x}\right)}{x\left(1+\dfrac{2}{x}\right)} = \lim_{x\to-\infty} \frac{3x}{x} \cdot \lim_{x\to-\infty} \frac{\left(1-\dfrac{5}{3x}\right)}{\left(1+\dfrac{2}{x}\right)} = \lim_{x\to-\infty} 3\cdot(1) = 3$$

The other end-behavior is also 3, after a very similar computation. Thus

$$\lim_{x\to-\infty} \frac{3x-5}{x+2} = \lim_{x\to\infty} \frac{3x-5}{x+2} = 3$$

Let us see the intercepts. If $x=0$, then clearly $\dfrac{3x-5}{x+2} = -\dfrac{5}{2}$. Thus the $y-$intercept is $\left(0,-\dfrac{5}{2}\right)$. For the $x-$intercept, we solve $\dfrac{3x-5}{x+2} = 0$.

$$\begin{aligned} \frac{3x-5}{x+2} &= 0 \qquad \text{multiply by } x+2 \\ 3x-5 &= 0 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

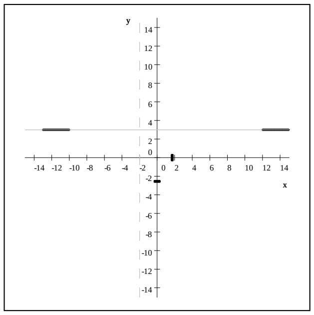
This should come as no surprise: a fraction can only be zero if its numerator is zero. Thus the $x-$intercept is $\left(\dfrac{5}{3},0\right)$.

If we now look at the numerator and denominator as factored polynomials (both linear) with their zeroes,

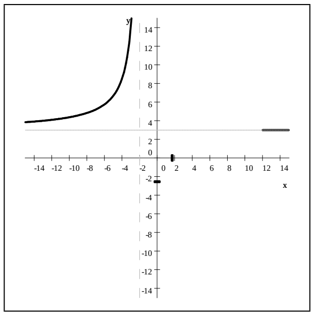
$$f(x) = \frac{3x-5}{x+2} = \frac{3\left(x-\dfrac{5}{3}\right)}{x+2}$$

we realize that f will change sign twice: once at $x=-2$ through a vertical asymptote and once at $x=\dfrac{5}{3}$ through a zero. Using this method, it is very important for us to understand that $x=\dfrac{5}{3}$ **is the only zero and** $x=-2$ **is the only discontinuity for** f. We are now ready to graph the function.

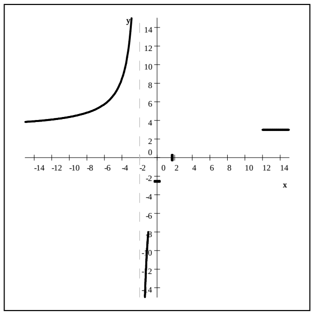
Step 1. We indicate the end-behavior, vertical asymptote and intercepts.



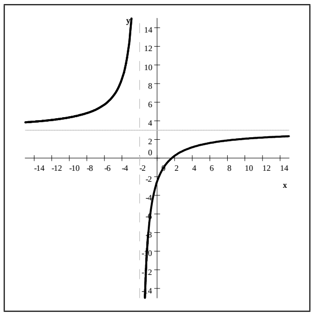
Step 2. We start graphing f, left to right. As we start at the left end-behavior and approach the vertical asymptote, we know that the graph will take very large (positive or negative) values near $x=-2$. From the end-behavior, we know that f is positive before $x=-2$. Because f does not have any zero or discontinuity before $x=-2$, f can only take large positive values before and near $x=-2$.



Step 3. Since $x+2$ in the denominator has an odd exponent (namely 1) f will change sign from the left of -2 to the right of -2. Because f is positive before -2, it will become negative after. So f takes large negative values after and near $x=-2$.



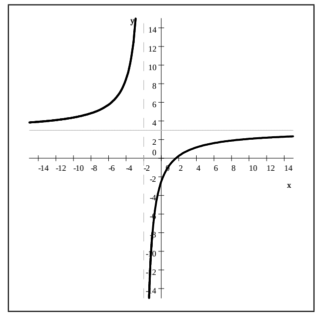
Step 4. Since $x-\frac{5}{3}$ in the numerator has an odd exponent (namely 1) f will change sign from the left of $\frac{5}{3}$ to the right of $\frac{5}{3}$. We connect the $x-$intercept with the end-behavior with a continuous line. The $y-$intercept helps us to draw the graph precisely.



Solution 2. $f(x)=\dfrac{3x-1}{x+2}$ is a rational function where both numerator and denominator are linear. If that's the case, the graph of the function is a "relative" of that of $y=\dfrac{1}{x}$. We simply divide the polynomials and bring the function to a form that helps us see what transformations are needed and in what order. $(3x-1)$ divided by $(x+2)$ is 3 with a reminder of -7. In other words,

$$\frac{3x-1}{x+2}=3-\frac{7}{x+2}=-7\cdot\frac{1}{x+2}+3$$

This means that we can graph f using transformations as follows.



Step 1. We graph $y = \dfrac{1}{x}$.

Step 2. We shift the graph to the left by 2 units

Step 3. We reflect the graph to the $x-$axis.

Step 4. We stretch the graph 7$-$fold along the $y-$axis.

Step 5. We shift the graph upward by 3 units.

4. Graph $f(x) = \dfrac{-(x+2)(x+1)^2(x-2)}{(x+1)x^2(x-1)(x-2)}$

Step 1. Simplify the equation of f.

We can cancel out linear factors if they appear both in the numerator and denominator. However, we need to do this carefully. Recall that two functions are equal if they have the same assignment and the **same domain**.

$$f(x) = \frac{-(x+2)(x+1)^2(x-2)}{(x+1)x^2(x-1)(x-2)} = \begin{cases} \dfrac{-(x+2)(x+1)}{x^2(x-1)} & \text{if} \quad x \neq -1, 2 \\ \text{undefined} & \text{if} \quad x = -1, 2 \end{cases}$$

The numbers that were zeroes of the denominator in the original function but not in the simplified one (in this case -1 and 2) are the **holes**. The numbers that are zeroes of the denominator of the simplified form (in this case 0 and 1) is where the graph will have a **vertical asymptote**.

Step 2. We will graph the simplified function, $g(x) = \dfrac{-(x+2)(x+1)}{x^2(x-1)}$

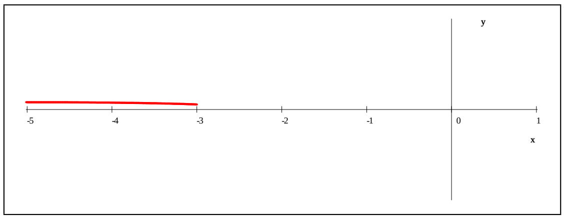
Step 2A. Determine the end-behavior of $g(x)$. As the computation below shows, the endbehavior is the same as that of the quotient of the leading terms of numerator and denominator, respectively.

$$\begin{aligned} \lim_{x\to\pm\infty} g(x) &= \lim_{x\to\pm\infty} \frac{-(x+2)(x+1)}{x^2(x-1)} = \lim_{x\to\pm\infty} \frac{-x^2-3x-2}{x^3-x} = \lim_{x\to\pm\infty} \frac{x^2\left(-1-\dfrac{3}{x}-\dfrac{2}{x^2}\right)}{x^3\left(1-\dfrac{1}{x^2}\right)} \\ &= \left(\lim_{x\to\pm\infty} \frac{x^2}{x^3}\right)\left(\lim_{x\to\pm\infty} \frac{-1-\dfrac{3}{x}-\dfrac{2}{x^2}}{1-\dfrac{1}{x^2}}\right) = 0(-1) = 0 \end{aligned}$$

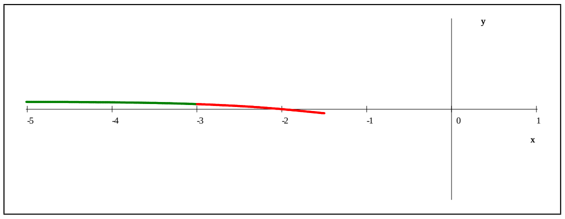
Step 2B. Graph the reduced fraction, $g(x) = \dfrac{-(x+2)(x+1)}{x^2(x-1)}$.

We will graph g left to right. The steps in graphing $g(x)$ involve the understanding what happens at negative infinity, -2, -1, 0, 1, and positive infinity.

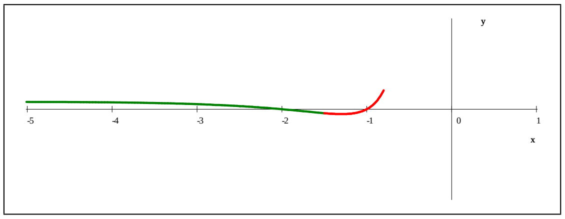
1. For large negative values of x: for all values of $x < -2$, all linear factors of numerator and denominator are negative. Since $g(x)$ has five linear factors and one additional multiplier of -1, $g(x)$ is positive for all $x < -2$. Since it is positive there, $g(x)$ approaches 0 from above for large negative values of x.



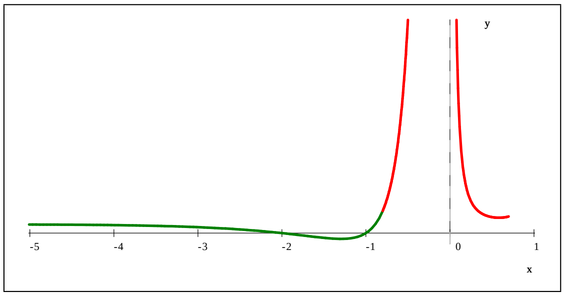
2. At $x = -2$, $g(x)$ changes sign in the numerator, from positive to negative. We know this because the linear factor that is zero at $x = -2$, $(x + 2)$ has an odd exponent, 1. Since this linear factor is in the numerator, the graph changes sign by passing through zero continuously.



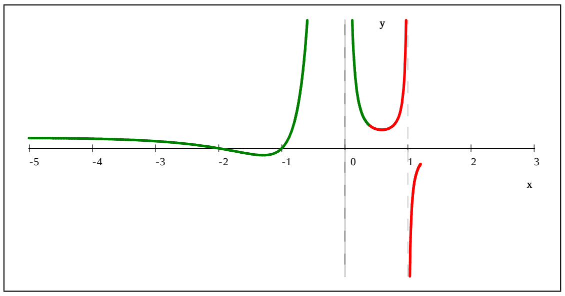
3. At $x = -1$, $g(x)$ changes sign in the numerator: the graph will pass through zero and becomes positive.



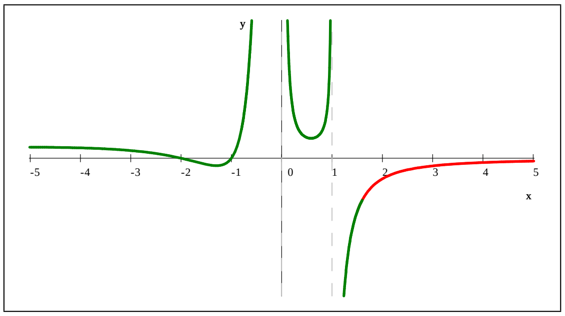
4. At $x = 0$, $g(x)$ does not change sign through a zero in the denominator: there is a vertical asymptote at 0 and the function values are large positive numbers on both sides of zero.



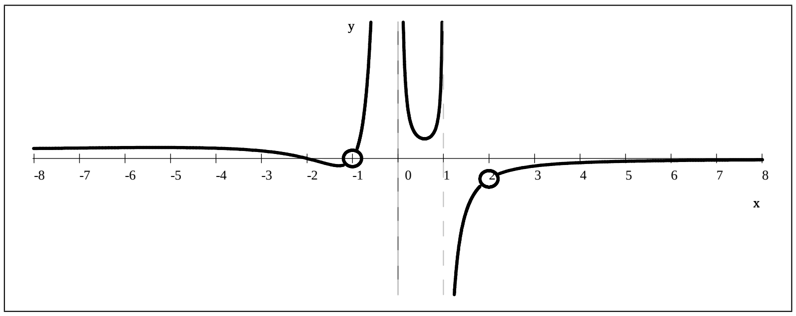
5. At $x = 1$, $g(x)$ changes sign in the denominator: : there is a vertical asymptote at 1 and the function values are large positive numbers the left and large negative numbers on the right of 1.



6. For large positive values of x: Since there is no more change in sign, the function will remain negative. Thus $g(x)$ approaches 0 from below for large positive values of x.



Step 3. We have now graphed $g(x)$. Our function has almost the same graph: f and g has slightly different domains. We just need to make sure that we do not include the two points that do not belong to the domain of f: at $x=-1$ and at $x=2$.



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