Definition: The **set of all natural numbers**, denoted by $\mathbb{N}$, is the infinite set

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

If we add two natural numbers, the sum is also a natural number. In other words, if x and y are natural numbers, then the sum $x + y$ is also a natural number. When this is true, we say that the set of all natural numbers is **closed under addition**. On the other hand, the set of all natural numbers is not closed under subtraction: while $10 - 3$ is a natural number, $3 - 10$ is not.

Theorem: *The set of all natural numbers is closed under addition and multiplication, but not under subtraction and division.*

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

Notice that the set of all integers contains all natural numbers. When this happens, we say that the set of all natural numbers is a subset of the set of all integers. Notation: $\mathbb{N} \subseteq \mathbb{Z}$.

Theorem: *The set of all integers is closed under addition, multiplication, and subtraction, but not under division.*

Definition: A number is **rational** if it can be written as a quotient of two integers.

For example, $\frac{3}{8}$ is a rational number because both 3 and 8 are integers and so $\frac{3}{8}$ is a quotient of two integers.

Definition: The **set of all rational numbers**, denoted by $\mathbb{Q}$, is the infinite set

$$\mathbb{Q} = \left\{ \frac{a}{b} : \ a \text{ and } b \text{ are integers, } b \neq 0 \right\}$$

Notice that the set of all rational numbers entirely contains the set of all integers, i.e. $\mathbb{Z} \subseteq \mathbb{Q}$. In fact, the three sets are such that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

For example, the number -5 is an integer and also a rational number, because we can write it as a quotient $\frac{-5}{1}$ where both -5 and 1 are integers. The number zero is also a rational number because it can be written as $\frac{0}{3}$.

Theorem: *The set of all rational numbers is closed under addition, multiplication, subtraction, and division.*

For some strange reason, mathematicians still needed more kind of numbers.

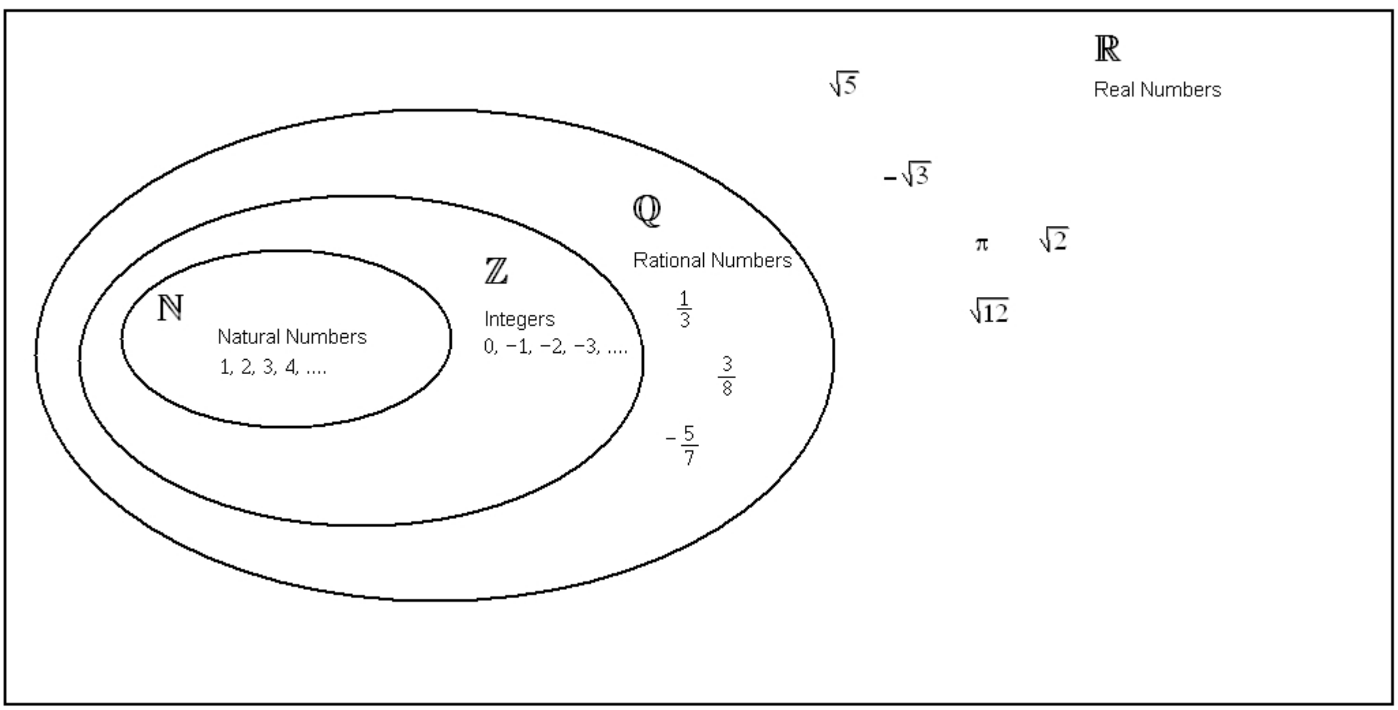
Definition: A number is **irrational** if it cannot be written as a quotient of two integers.

This is a very strange property because there are so many different integers from which to choose. However, irrational numbers exist. For example, π and $\sqrt{2}$ are irrational numbers. Surprisingly, in a sense, there are many more irrational numbers than rational numbers. (In a fascinating subject within mathematics called set theory, mathematicians have developed language to compare infinite sets. In that comparison, the set of irrational numbers proved to be much greater than the set of rational numbers.)

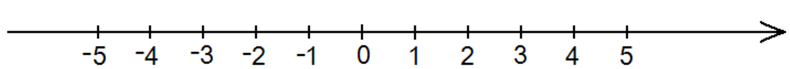
Definition: The set of all **real numbers**, denoted by $\mathbb{R}$, is the collection of all rational and irrational numbers.

The set of all real numbers contain all previous number sets as a subset. For example, every rational number is a real number.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



Mathematicians proved that there are exactly as many real nubers as many points there are on a straight line. Given a line, for every point on it, we can uniquely assign a real number to that point. Vice versa, for every real number, there is exactly one point on the line. This correspondance is expressed by the concept of the **number line**.



Theorem: *Every terminating decimal represents a rational number.*

We have proved this in class. To convert a terminating decimal to a fraction of integers, see the handout Fractions and Decimals.

Theorem: *Every non-terminating, repeating decimal represents a rational number.*

We have proved this in class. To convert a repeating decimal to a fraction of integers, see the handout Fractions and Decimals.

There is an important conclusion that can be drawn from these two facts. Consider $\sqrt{2}$, for example. If we accept the fact that $\sqrt{2}$ is irrational (which can be proved at this level) then it follows that its decimal presentation can not be terminating. (Why not?) And also, the decimal presentation of $\sqrt{2}$ can not be repeating. What is left for poor irrational numbers?

Theorem: *The decimal presentation of irrational numbers is non-terminating and non-repeating.*

By definition of irrational numbers, $\sqrt{2}$ can not be expressed as a fraction of two integers. We have just seen that $\sqrt{2}$ can not really be written as a decimal. If we attempted to write $\sqrt{2}$ as a decimal, we can only write approximations of the number, and never the exact value.

When we are prompted to give a number's *exact value*, in case of $\sqrt{2}$, the symbol $\sqrt{2}$ is our only option. Fractions formed from integers such as $\frac{141}{100}$ or decimals such as 1.41 are only *approximations*.