Carl Friedrich Gauss (1777-1855) is probably one of the greatest mathematicians of all time. He made major contributions to most areas within mathematics. The following story is from when Gauss was still at primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple.

Here is the question: $1+2+3+...+100=x$

We repeat this line, but this time, backward: $100+99+98+...+1=x$

And we will add the two lines - the result will be twice the desired sum. Instead of adding row 1 and then row 2, we will add the numbers column by column. The sum in the first column is $1+100=101$.

$$\begin{array}{rcrcrcccrcr} & 1 & + & 2 & + & 3 & + & .... & + & 100 & = & x \\ + & 100 & + & 99 & + & 98 & + & ... & + & 1 & = & x \\ \hline & 101 & + & 101 & + & 101 & + & ... & + & 101 & = & 2x \end{array}$$

The sum of the second column is $2+99=101$. The sum of the third column is $3+98=101$. And so on, the sum of each column is 101 because as we step to the right, the number in the first row increase by 1 and the number in the second row decrease by 1. Thus the sum remains 101.

$$\begin{array}{rcrcrcccrcr} & 1 & + & 2 & + & 3 & + & .... & + & 100 & = & x \\ + & 100 & + & 99 & + & 98 & + & ... & + & 1 & = & x \\ \hline & 101 & + & 101 & + & 101 & + & ... & + & 101 & = & 2x \end{array}$$

When we add the same number to itself repeatedly, that can be re-written as multiplication. We have 100 columns, so we added 101 to itself 100 times. So the long sum can be replaced by a single multiplication.

$$\begin{aligned} 100\cdot 101 &= 2x \\ 10\,100 &= 2x \\ 5050 &= x \end{aligned}$$

This method can be applied to long sums, as long as the numbers increase or decrease by the same amount.

**Example 1.** Find the sum $3+6+9+12+....+600$.

**Solution:** We will apply Gauss's method.

$$\begin{array}{ccccccccc} 3 & + & 6 & + & 9 & + & ... & + & 600 & = & x \\ 600 & + & 597 & + & 594 & + & ... & + & 3 & = & x \end{array}$$

The easy question is: what is the sum in each column? Clearly 603. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 3 & + & 6 & + & 9 & + & ... & + & 600 & = & x \end{array}$$

The last number will have to get the label 200. So, we have 200 columns and each of them adds up to 603.

$$\begin{aligned} 603+603+603+...+603 &= 2x \\ 200\cdot 603 &= 2x \\ 120\,600 &= 2x \\ 60\,300 &= x \end{aligned}$$

So the sum $3+6+9+...+600=\boxed{60\,300}$.

**Example 2.** Find the sum $31 + 38 + 45 + .... + 423$.

**Solution:** We will apply Gauss's method.

$$\begin{array}{ccccccccccc} 31 & + & 38 & + & 45 & + & . \; . \; . & + & 423 & = & x \\ 423 & + & 416 & + & 409 & + & . \; . \; . & + & 31 & = & x \end{array}$$

The easy question is: what is the sum in each column? Clearly 454. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 31 & + & 38 & + & 45 & + & . \; . \; . & + & 423 & = & x \end{array}$$

We can see that as the label increases by 1, the number in the first row increases by 7. Because of this, we might suspect that the number belonging to label $n$ has to do with $7n$. Let us try: If $n = 1$, then $7 \cdot 1 + b = 31$ gives us $b = 24$. Thus the connection between the label and the number with that label is $M = 7n + 24$.

Let's try this formula for $n = 2$. If $n = 2$, then $M = 7 \cdot 2 + 24 = 38$.

Let's try this for $n = 3$. If $n = 3$, then $M = 7 \cdot 3 + 24 = 45$.

And so on, our formula works. So what label $n$ belongs to the number 423? To find out, we just need to solve the equation $423 = 7n + 24$.

$$\begin{aligned} 423 &= 7n + 24 \\ 399 &= 7n \\ 57 &= n \end{aligned}$$

Thus, there are 57 columns.
The rest is easy:

$$\begin{aligned} 454 + 454 + 454 + ... + 454 &= 2x \\ 57 \cdot 454 &= 2x \\ 25\,878 &= 2x \\ 12\,939 &= x \end{aligned}$$

So the sum $31 + 38 + 45 + .... + 423 = \boxed{12\,939}$

## Practice Problems

Compute each of the following sums.

1. $19 + 30 + 41 + .... + 4078$
2. $29 + 34 + 39 + ... + 374$
3. $15 + 17 + 19 + ... + 643$
4. $32 + 41 + 50 + ... + 698$
5. $11 + 17 + 23 + ... + 1325$
6. $16 + 23 + 30 + ... + 1094$
7. $20 + 27 + 34 + ... + 783$
8. $31 + 37 + 43 + ... + 1015$
9. $\dfrac{1}{2019} + \dfrac{2}{2019} + \dfrac{3}{2019} + ... + \dfrac{2018}{2019}$

10. What is the sum of all numbers shown in the square?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |

# Answers

1. 757 945
2. 14 105
3. 103 635
4. 27 375
5. 14 960
6. 86 025
7. 44 165
8. 86295
9. 1009
10. 1000



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