The following trigonometric equations can be solved using the double-angle formulas for sine, cosine, and tangent.

Sample Problems

Solve each of the following equations.

1. $\sin 2x = \cos x$
2. $-\cos 2x = \sin x$
3. $\cos 2x = \cos x$
4. $\cos 2x + 5\cos x = -3$
5. $\sin x + \sin 2x = \tan x$
6. $\cos x + \dfrac{\sin^2 x}{\cos x} + \sin x + \sin 2x = \dfrac{1}{\cos x}$
7. $\tan 2x = \tan x$

Sample Problems - Answers

1. $x = \dfrac{\pi}{2} + k_1\pi$ or $x = \dfrac{\pi}{6} + 2k_2\pi$ or $x = \dfrac{5\pi}{6} + 2k_3\pi$ where $k_1, k_2, k_3 \in \mathbb{Z}$
2. $x = \dfrac{7\pi}{6} + 2k\pi$ or $x = -\dfrac{\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$ or $x = \dfrac{\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$
3. $x = \pm\dfrac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$ or $x = 2k\pi$ where $k \in \mathbb{Z}$
4. $x = \pm\dfrac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$
5. $x = \pm\dfrac{\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$ or $x = k\pi$ where $k \in \mathbb{Z}$
6. $x = k\pi$ where $k \in \mathbb{Z}$ or $x = \pm\dfrac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$
7. $x = k\pi$ where $k \in \mathbb{Z}$

Sample Problems - Solutions

1. $\sin 2x = \cos x$

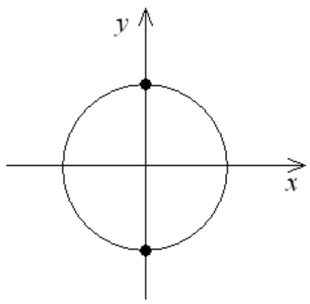
Solution: We will start by re-writing $\sin 2x$ using the double-angle formula for sine.

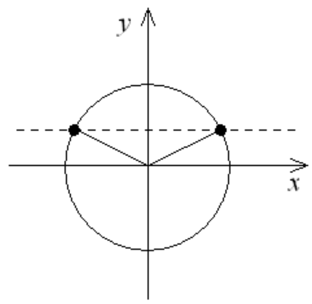
$$\begin{aligned} \sin 2x &= \cos x \\ 2\sin x\cos x &= \cos x && \text{subtract } \cos x \\ 2\sin x\cos x - \cos x &= 0 && \text{factor out } \cos x \\ \cos x\,(2\sin x - 1) &= 0 \end{aligned}$$

Now we use the zero-product rule.

$$\cos x = 0 \qquad \text{or} \qquad 2\sin x - 1 = 0$$

$$\cos x = 0 \qquad\qquad \sin x = \frac{1}{2}$$





$$x = \frac{\pi}{2} + k\pi \text{ where } k \in \mathbb{Z} \qquad x = \frac{\pi}{6} + 2k\pi \text{ or } x = \frac{5\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z}$$

We should check. We will check the values $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$. All other solution is co-terminal to one of these.

If $x = \frac{\pi}{2}$, then

$$\text{LHS} = \sin\left(2 \cdot \frac{\pi}{2}\right) = \sin \pi = 0 \text{ and } \text{RHS} = \cos\frac{\pi}{2} = 0 \text{ - this is a solution}$$

If $x = \frac{3\pi}{2}$, then

$$\text{LHS} = \sin\left(2 \cdot \frac{3\pi}{2}\right) = \sin 3\pi = 0 \text{ and } \text{RHS} = \cos\frac{3\pi}{2} = 0 \text{ - this is a solution}$$

If $x = \frac{\pi}{6}$, then

$$\text{LHS} = \sin\left(2 \cdot \frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \text{ and } \text{RHS} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \text{ - this is a solution}$$

If $x = \frac{5\pi}{6}$, then

$$\text{LHS} = \sin\left(2 \cdot \frac{5\pi}{6}\right) = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \text{ and } \text{RHS} = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \text{ - this is a solution}$$

So the final answer is $\boxed{x = \frac{\pi}{2} + k_1\pi \text{ or } x = \frac{\pi}{6} + 2k_2\pi \text{ or } x = \frac{5\pi}{6} + 2k_3\pi \text{ where } k_1, k_2, k_3 \in \mathbb{Z}}$

2. $-\cos 2x = \sin x$

Solution: We will start by re-writing $\cos 2x$ using the double-angle formula for cosine. This formula has three forms; we will use the one that expresses $\cos 2x$ in terms of $\sin x$, because $\sin x$ is present on the other side of the equation.

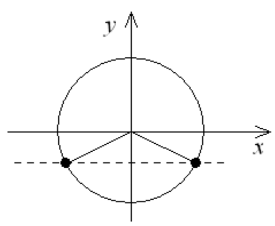
$$\begin{aligned} -\cos 2x &= \sin x \\ -\left(1 - 2\sin^2 x\right) &= \sin x \\ -1 + 2\sin^2 x &= \sin x \end{aligned}$$

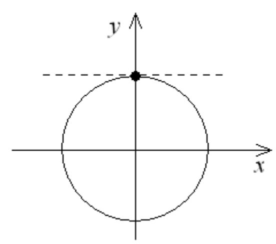
This equation is quadratic in $\sin x$ and so we will first solve it for $\sin x$. We reduce one side to zero and factor.

$$\begin{aligned} 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \end{aligned}$$

$2\sin x + 1 = 0$ or $\sin x - 1 = 0$

$\sin x = -\dfrac{1}{2}$ $\qquad$ $\sin x = 1$





$$\boxed{x = \frac{7\pi}{6} + 2k\pi \text{ or } x = -\frac{\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z} \quad \text{or} \quad x = \frac{\pi}{2} + 2k\pi \text{ where } k \in \mathbb{Z}}$$

As always, we should check our solution. We will omit that here and leave it to the reader.

3. $\cos 2x = \cos x$

Solution: We will start by re-writing $\cos 2x$ using the double-angle formula for cosine. This formula has three forms; we will use the one that expresses $\cos 2x$ in terms of $\cos x$, because that is present on the other side of the equation.

$$\begin{aligned} \cos 2x &= \cos x \\ 2\cos^2 x - 1 &= \cos x \end{aligned}$$

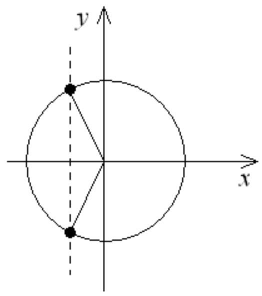
This equation is quadratic in $\cos x$ and so we will first solve it for $\cos x$. We reduce one side to zero and factor.

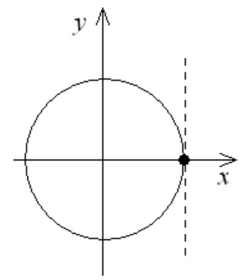
$$\begin{aligned} 2\cos^2 x - \cos x - 1 &= 0 && \text{factor} \\ (2\cos x + 1)(\cos x - 1) &= 0 \end{aligned}$$

$2\cos x + 1 = 0$ or $\cos x - 1 = 0$

$\cos x = -\dfrac{1}{2}$ $\qquad$ $\cos x = 1$

$\cos x = -\dfrac{1}{2}$ $\qquad\qquad$ $\cos x = 1$





$$\boxed{x = \pm\frac{2\pi}{3} + 2k\pi \quad \text{where } k \in \mathbb{Z} \qquad \text{or} \qquad x = 2k\pi \quad \text{where } k \in \mathbb{Z}}$$

As always, we should check our solution. We will omit that here and leave it to the reader.

4. $\cos 2x + 5\cos x = -3$

Solution: We will start by re-writing $\cos 2x$ using the double-angle formula for cosine. This formula has three forms; we will use the one that expresses $\cos 2x$ in terms of $\cos x$, because that is present in the rest of the equation.

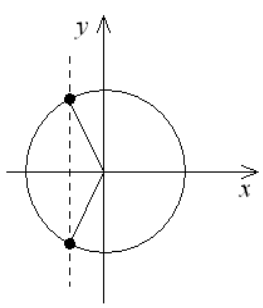
$$\begin{aligned} \cos 2x + 5\cos x &= -3 \\ 2\cos^2 x - 1 + 5\cos x &= -3 \end{aligned}$$

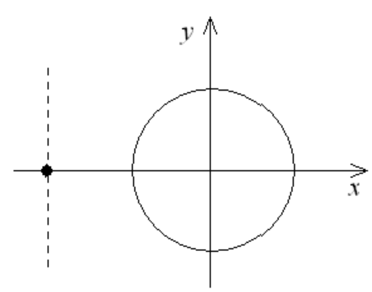
This equation is quadratic in $\cos x$ and so we will first solve it for $\cos x$. We reduce one side to zero and factor.

$$\begin{aligned} 2\cos^2 x + 5\cos x + 2 &= 0 \qquad \text{factor} \\ (2\cos x + 1)(\cos x + 2) &= 0 \end{aligned}$$

$2\cos x + 1 = 0$ $\qquad$ or $\qquad$ $\cos x + 2 = 0$

$\cos x = -\dfrac{1}{2}$ $\qquad\qquad$ $\cos x = -2$





$x = \pm\dfrac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$ $\qquad$ or $\qquad$ there is no number with $\cos x = -2$

So the final answer is $\boxed{x = \pm\dfrac{2\pi}{3} + 2k\pi \quad \text{where } k \in \mathbb{Z}}$.

As always, we should check our solution. We will omit that here and leave it to the reader.

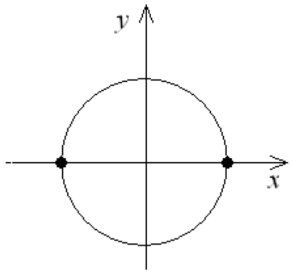
5. $\sin x + \sin 2x = \tan x$

Solution: We first re-write everything in terms of $\sin x$ and $\cos x$.

$$\begin{aligned} \sin x + \sin 2x &= \tan x \\ \sin x + 2\sin x \cos x &= \frac{\sin x}{\cos x} \\ \sin x + 2\sin x \cos x - \frac{\sin x}{\cos x} &= 0 && \text{factor out } \sin x \\ \sin x \left(1 + 2\cos x - \frac{1}{\cos x}\right) &= 0 \end{aligned}$$

$$\sin x = 0 \quad \text{or} \quad 1 + 2\cos x - \frac{1}{\cos x} = 0$$

Case 1. If $\sin x = 0$, then



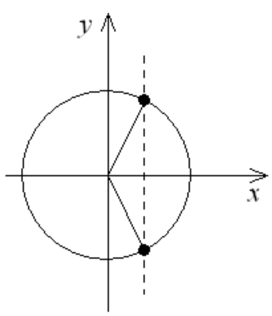
$x = k\pi$ where $k \in \mathbb{Z}$.

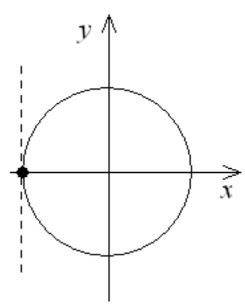
Case 2. If $1 + 2\cos x - \dfrac{1}{\cos x} = 0$. We will multiply both sides by $\cos x$.

$$\begin{aligned} 1 + 2\cos x - \frac{1}{\cos x} &= 0 && \text{multiply by } \cos x \\ \cos x + 2\cos^2 x - 1 &= 0 \\ 2\cos^2 x + \cos x - 1 &= 0 \\ (2\cos x - 1)(\cos x + 1) &= 0 \end{aligned}$$

$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \dfrac{1}{2}$ $\qquad$ $\cos x = -1$





$x = \pm\dfrac{\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$ or $x = \pi + 2k\pi$ where $k \in \mathbb{Z}$

So the solution, after considering all the cases, is $\boxed{x = \pm\dfrac{\pi}{3} + 2k\pi \text{ where } k \in \mathbb{Z} \text{ or } x = k\pi \text{ where } k \in \mathbb{Z}}$.

As always, we should check our solution. We will omit that here and leave it to the reader.

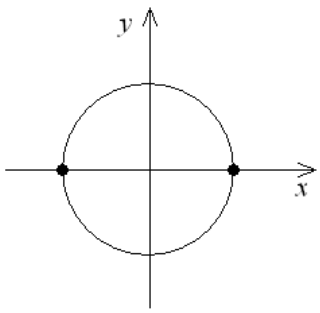
6. $\cos x + \dfrac{\sin^2 x}{\cos x} + \sin x + \sin 2x = \dfrac{1}{\cos x}$

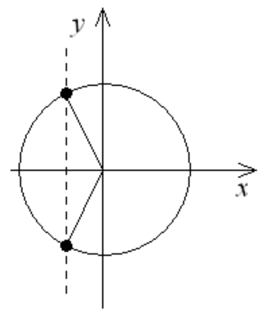
Solution: Let us notice that $\cos x \neq 0$. Knowing the domain of both sides of an equation might be useful later.

$$\begin{aligned}
\cos x + \frac{\sin^2 x}{\cos x} + \sin x + \sin 2x &= \frac{1}{\cos x} \\
\cos x + \frac{\sin^2 x}{\cos x} + \sin x + 2\sin x\cos x &= \frac{1}{\cos x} && \cos x = \frac{\cos^2 x}{\cos x} \\
\frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} + \sin x + 2\sin x\cos x &= \frac{1}{\cos x} \\
\frac{\cos^2 x + \sin^2 x}{\cos x} + \sin x + 2\sin x\cos x &= \frac{1}{\cos x} && \cos^2 x + \sin^2 x = 1 \\
\frac{1}{\cos x} + \sin x + 2\sin x\cos x &= \frac{1}{\cos x} && \text{subtract } \frac{1}{\cos x} \\
\sin x + 2\sin x\cos x &= 0 && \text{factor out } \sin x \\
\sin x\,(1 + 2\cos x) &= 0
\end{aligned}$$

$\sin x = 0$ or $1 + 2\cos x = 0$

$\cos x = -\dfrac{1}{2}$





$x = k\pi$ where $k \in \mathbb{Z}$ or $x = \pm\dfrac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$

So the solution is $\boxed{x = k\pi \text{ where } k \in \mathbb{Z} \text{ or } x = \pm\frac{2\pi}{3} + 2k\pi \text{ where } k \in \mathbb{Z}}$

As always, we should check our solution. We will omit that here and leave it to the reader.

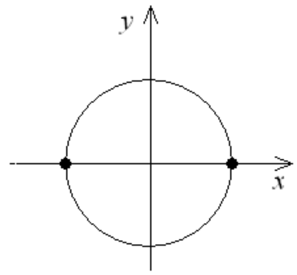
7. $\tan 2x = \tan x$

Solution:

$$\begin{aligned}
\tan 2x &= \tan x \\
\frac{2\tan x}{1 - \tan^2 x} &= \tan x \\
\frac{2\tan x}{1 - \tan^2 x} - \tan x &= 0 \\
\tan x \left(\frac{2}{1 - \tan^2 x} - 1\right) &= 0
\end{aligned}$$

$$\tan x \left(\frac{2}{1-\tan^2 x} - 1 \right) = 0$$

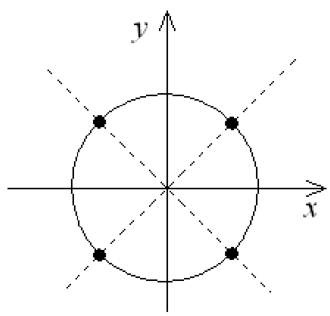
Case 1. If $\tan x = 0$, then



$$x = k\pi \quad \text{where} \quad k \in \mathbb{Z}$$

Case 2. If $\dfrac{2}{1-\tan^2 x} - 1 = 0$

$$\begin{aligned} \frac{2}{1-\tan^2 x} - 1 &= 0 \\ \frac{2}{1-\tan^2 x} &= 1 \qquad \text{multiply by } 1-\tan^2 x \\ 2 &= 1-\tan^2 x \\ \tan^2 x &= 1 \\ \tan x &= \pm 1 \end{aligned}$$



$$x = \pm\frac{\pi}{4} + k\pi \text{ where } k \in \mathbb{Z}$$

But if $\tan^2 x = 1$, then the expression $\dfrac{2}{1-\tan^2 x}$ is undefined, so these might not be solutions. Let us be careful and check using the equation given. Recall the original equation $\tan 2x = \tan x$.
If $x = \dfrac{\pi}{4}$, then the left-hand side is

$$\text{LHS} = \tan\left(2 \cdot \frac{\pi}{4}\right) = \tan\frac{\pi}{2} = \text{undefined}$$

and so $x = \dfrac{\pi}{4}$ is not a solution. It turns out that all four points on the unit circle shown on the picture above give rise to numbers that do not work with the original equation. So the final answer is just $x = k\pi$ where $k \in \mathbb{Z}$. We should check that these work. Indeed, if $x = 0$, then

$$\text{LHS} = \tan(2 \cdot 0) = \tan 0 = 0 \text{ and RHS} = \tan 0 = 0$$

and if $x = \pi$, then

$$\text{LHS} = \tan(2 \cdot \pi) = \tan 2\pi = 0 \text{ and RHS} = \tan \pi = 0$$

So the final answer is $\boxed{x = k\pi \text{ where } k \in \mathbb{Z}}$.

Please note that this problem can be solved using other methods. The following is a much more elegant solution. The method is based on understanding what it takes for two angles to have the same tangent: they must differ by a multiple of π. In short, if $\tan A = \tan B$ then $A = B + k\pi$ for some integer k.

$$\begin{aligned} \tan 2x &= \tan x \\ 2x &= x + k\pi \qquad \text{subtract } x \\ x &= k\pi \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

