Sample Problems

Solve each of the following equations.

1. $\sin x + \cos x = -1$
2. $\sin x - \sqrt{3}\cos x = 1$

The following problem is not an equation but it can be solved using the same technique and it is important enough to be shown here.

3. Find the smallest and largest values of the function $f(x) = 2\cos x + 3\sin x$.

Practice Problems

Solve each of the following equations.

1. $\sin x - \cos x = 1$
2. $\sqrt{3}\sin x + \cos x = -1$
3. Find the smallest and greatest values of each of the following functions.
   a) $f(x) = 3\sin x - 4\cos x$
   b) $g(x) = \dfrac{1}{2}\sin x + 2\cos x$

Sample Problems - Answers

1. $x = \pi + 2k_1\pi$ or $x = -\dfrac{\pi}{2} + 2k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$
2. $x = \dfrac{\pi}{2} + 2k_1\pi$ or $x = \dfrac{7\pi}{6} + 2k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$
3. smallest value: $-\sqrt{13}$ greatest value: $\sqrt{13}$

Practice Problems - Answers

1. $x = \dfrac{\pi}{2} + 2k_1\pi$ or $x = \pi + 2k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$
2. $x = \pi + 2k_1\pi$ or $x = -\dfrac{\pi}{3} + 2k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$
3. a) smallest value: -5 greatest value: 5

   b) smallest value: $-\sqrt{4.25}$ greatest value: $\sqrt{4.25}$

Sample Problems - Solutions

1. $\sin x + \cos x = -1$

   Solution: We first scale down the equation so that the coefficients of $\sin x$ and $\cos x$ become trigonometric function values of the same angle.

   The coefficient of $\sin x$ and $\cos x$ are both 1 and $\sqrt{1^2 + 1^2} = \sqrt{2}$, so we divide both sides by $\sqrt{2}$

$$\begin{aligned} \sin x + \cos x &= -1 \qquad \text{divide by } \sqrt{2} \\ \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x &= -\frac{1}{\sqrt{2}} \end{aligned}$$

   We now re-write these coefficients as trigonometric function values of the same angle. We actually have two options that work:

   Option 1. We re-write $\dfrac{1}{\sqrt{2}}$ as first $\sin\left(\dfrac{\pi}{4}\right)$ and then as $\cos\left(\dfrac{\pi}{4}\right)$

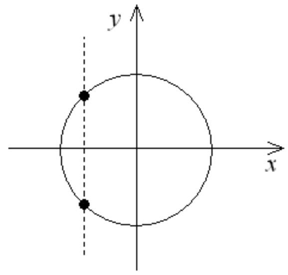
$$\begin{aligned} \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x &= -\frac{1}{\sqrt{2}} \\ \sin\left(\frac{\pi}{4}\right)\sin x + \cos\left(\frac{\pi}{4}\right)\cos x &= -\frac{1}{\sqrt{2}} \end{aligned}$$

   We can now realize the difference formula for cosine on the left-hand side of the equation.

$$\begin{aligned} \cos\left(\frac{\pi}{4}\right)\cos x + \sin\left(\frac{\pi}{4}\right)\sin x &= -\frac{1}{\sqrt{2}} \\ \cos\left(x - \frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \end{aligned}$$

   We now solve for $x - \dfrac{\pi}{4}$ and then for x

$$\cos\left(x-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$



$$x-\frac{\pi}{4}=\pm\frac{3\pi}{4}+2k\pi \quad \text{where } k\in\mathbb{Z}$$

We simply add $\frac{\pi}{4}$ to both sides to solve for x

$$x=\frac{\pi}{4}\pm\frac{3\pi}{4}+2k\pi \quad \text{where } k\in\mathbb{Z}$$

$$\begin{aligned} x &= \frac{\pi}{4}+\frac{3\pi}{4}+2k_1\pi \quad \text{where } k_1\in\mathbb{Z} \quad \text{or} \quad x=\frac{\pi}{4}-\frac{3\pi}{4}+2k_2\pi \quad \text{where } k_2\in\mathbb{Z} \\ x &= \pi+2k_1\pi \quad \text{where } k_1\in\mathbb{Z} \quad \text{or} \quad x=-\frac{\pi}{2}+2k_2\pi \quad \text{where } k_2\in\mathbb{Z} \end{aligned}$$

We check: If x is co-terminal to π, then $\sin x+\cos x=0+(-1)=-1$, and if x is co-terminal to $-\frac{\pi}{2}$, then $\sin x+\cos x=-1+0=-1$ and so all of these solutions are correct.

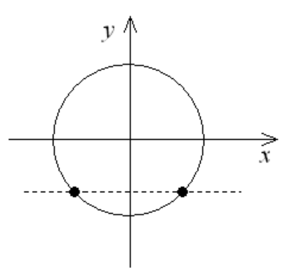
Option 2. We re-write $\frac{1}{\sqrt{2}}$ as first $\cos\left(\frac{\pi}{4}\right)$ and then as $\sin\left(\frac{\pi}{4}\right)$

$$\begin{aligned} \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x &= -\frac{1}{\sqrt{2}} \\ \cos\left(\frac{\pi}{4}\right)\sin x+\sin\left(\frac{\pi}{4}\right)\cos x &= -\frac{1}{\sqrt{2}} \end{aligned}$$

We can now realize the sum formula for sine on the left-hand side of the equation.

$$\begin{aligned} \sin x\cos\left(\frac{\pi}{4}\right)+\cos x\sin\left(\frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \\ \sin\left(x+\frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \end{aligned}$$

We now solve for $x+\frac{\pi}{4}$ and then for x



$$x + \frac{\pi}{4} = -\frac{\pi}{4} + 2k_1\pi \text{ where } k_1 \in \mathbb{Z} \quad \text{or} \quad x + \frac{\pi}{4} = -\frac{3\pi}{4} + 2k_2\pi \text{ where } k_2 \in \mathbb{Z}$$

We subtract $\frac{\pi}{4}$ from both sides to solve for x.

$$\begin{aligned} x &= -\frac{\pi}{4} - \frac{\pi}{4} + 2k_1\pi \text{ where } k_1 \in \mathbb{Z} \quad \text{or} \quad x = -\frac{\pi}{4} - \frac{3\pi}{4} + 2k_2\pi \quad \text{where } k_2 \in \mathbb{Z} \\ x &= -\frac{\pi}{2} + 2k_1\pi \text{ where } k_1 \in \mathbb{Z} \quad \text{or} \quad x = -\pi + 2k_2\pi \quad \text{where } k_2 \in \mathbb{Z} \end{aligned}$$

We check: If x is co-terminal to $-\pi$, then $\sin x + \cos x = 0 + (-1) = -1$, and if x is co-terminal to $-\frac{\pi}{2}$, then $\sin x + \cos x = -1 + 0 = -1$ and so all of these solutions are correct.

2. $\sin x - \sqrt{3} \cos x = 1$

Solution: We first scale down the equation so that the coefficients of $\sin x$ and $\cos x$ become trigonometric function values of the same angle.

The coefficient of $\sin x$ and $\cos x$ are both 1 and $\sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$, so we divide both sides by 2.

$$\begin{aligned} \sin x - \sqrt{3} \cos x &= 1 \qquad \text{divide by 2} \\ \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x &= \frac{1}{2} \end{aligned}$$

We now re-write these coefficients as trigonometric function values of the same angle. We actually have two options that work:

Option 1. We re-write $\frac{1}{2}$ as first $\cos\left(\frac{\pi}{3}\right)$ and $\frac{\sqrt{3}}{2}$ as $\sin\left(\frac{\pi}{3}\right)$

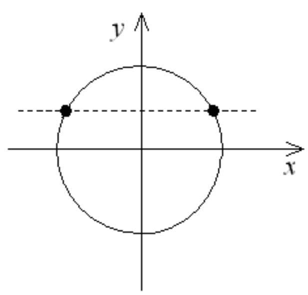
$$\begin{aligned} \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x &= \frac{1}{2} \\ \cos\left(\frac{\pi}{3}\right) \sin x - \sin\left(\frac{\pi}{3}\right) \cos x &= \frac{1}{2} \end{aligned}$$

We can now realize the difference formula for sine on the left-hand side of the equation.

$$\begin{aligned} \sin x \cos\left(\frac{\pi}{3}\right) - \cos x \sin\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ \sin\left(x - \frac{\pi}{3}\right) &= \frac{1}{2} \end{aligned}$$

We now solve for $x - \frac{\pi}{3}$ and then for x

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$$



$$x - \frac{\pi}{3} = \frac{\pi}{6} + 2k_1\pi \quad \text{where } k_1 \in \mathbb{Z} \quad \text{or} \quad x - \frac{\pi}{3} = \frac{5\pi}{6} + 2k_2\pi \quad \text{where } k_2 \in \mathbb{Z}$$

We add $\frac{\pi}{3}$ to both sides to solve for x

$$\begin{aligned} x &= \frac{\pi}{3} + \frac{\pi}{6} + 2k_1\pi \quad \text{where } k_1 \in \mathbb{Z} \quad \text{or} \quad x = \frac{\pi}{3} + \frac{5\pi}{6} + 2k_2\pi \quad \text{where } k_2 \in \mathbb{Z} \\ x &= \frac{\pi}{2} + 2k_1\pi \quad \text{where } k_1 \in \mathbb{Z} \quad \text{or} \quad x = \frac{7\pi}{6} + 2k_2\pi \quad \text{where } k_2 \in \mathbb{Z} \end{aligned}$$

We check: If x is co-terminal to $\frac{\pi}{2}$, then $\sin x - \sqrt{3}\cos x = 1 - 0 = 1$, and if x is co-terminal to $\frac{7\pi}{6}$, then $\sin x - \sqrt{3}\cos x = \sin\left(\frac{7\pi}{6}\right) - \sqrt{3}\cos\left(\frac{7\pi}{6}\right) = -\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + \frac{3}{2} = 1$ and so all of these solutions are correct.

Option 2. We re-write $\frac{1}{2}$ as first $\sin\left(\frac{\pi}{6}\right)$ and $\frac{\sqrt{3}}{2}$ as $\cos\left(\frac{\pi}{6}\right)$

$$\begin{aligned} \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x &= \frac{1}{2} \\ \sin\left(\frac{\pi}{6}\right)\sin x - \cos\left(\frac{\pi}{6}\right)\cos x &= \frac{1}{2} \end{aligned}$$

This is almost the sum formula for cosine, we are just off by a negative sign.

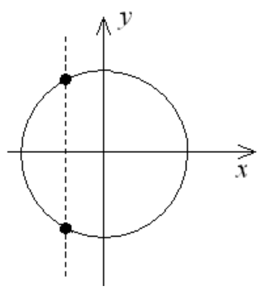
$$\begin{aligned} -\left(\cos x \cos\left(\frac{\pi}{6}\right) - \sin x \sin\left(\frac{\pi}{6}\right)\right) &= \frac{1}{2} \\ \cos x \cos\left(\frac{\pi}{6}\right) - \sin x \sin\left(\frac{\pi}{6}\right) &= -\frac{1}{2} \end{aligned}$$

We can now realize the difference formula for cosine on the left-hand side of the equation.

$$\cos\left(x + \frac{\pi}{6}\right) = -\frac{1}{2}$$

We now solve for $x + \frac{\pi}{6}$ and then for x

$$\cos\left(x + \frac{\pi}{6}\right) = -\frac{1}{2}$$



$$x + \frac{\pi}{6} = \pm\frac{2\pi}{3} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

We subtract $\frac{\pi}{6}$ from both sides to solve for x

$$x = -\frac{\pi}{6} \pm \frac{2\pi}{3} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$x = -\frac{\pi}{6}+\frac{2\pi}{3}+2k_1\pi \quad \text{where } k_1\in\mathbb{Z} \quad \text{or} \quad x=-\frac{\pi}{6}-\frac{2\pi}{3}+2k_2\pi \quad \text{where } k_2\in\mathbb{Z}$$
$$x = \frac{\pi}{2}+2k_1\pi \quad \text{where } k_1\in\mathbb{Z} \quad \text{or} \quad x=-\frac{5\pi}{6}+2k_2\pi \quad \text{where } k_2\in\mathbb{Z}$$

We check: If x is co-terminal to $\frac{\pi}{2}$, then $\sin x-\sqrt{3}\cos x=1-0=1$, and if x is co-terminal to $-\frac{5\pi}{6}$, then $\sin x-\sqrt{3}\cos x=\sin\left(-\frac{5\pi}{6}\right)-\sqrt{3}\cos\left(-\frac{5\pi}{6}\right)=-\frac{1}{2}-\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{1}{2}+\frac{3}{2}=1$ and so all of these solutions are correct.

3. Find the smallest and largest values of the function $f(x)=2\cos x+3\sin x$.

Solution: Just as before, we will be able to write $2\cos x+3\sin x$ as a single trignonmetric function. First, we scale down the coefficients so that they can be realized as sine and cosine of the same angle. $\sqrt{2^2+3^2}=\sqrt{13}$ and so we will go for $\frac{2}{\sqrt{13}}$ and $\frac{3}{\sqrt{13}}$ as the sine and cosine of the same angle θ. However, we cannot divide both sides by $\sqrt{13}$ - there is just one side. So, we will do the next best thing, factoring.

$$f(x)=2\cos x+3\sin x=1\cdot(2\cos x+3\sin x)=\frac{\sqrt{13}}{\sqrt{13}}\cdot(2\cos x+3\sin x)=\sqrt{13}\left(\frac{2}{\sqrt{13}}\cos x+\frac{3}{\sqrt{13}}\sin x\right)$$

This technique might seem strange at first, but we can easily apply the distributive law to see that we did not change the value of $f(x)$. Let θ be an angle with $\sin\theta=\frac{2}{\sqrt{13}}$ and $\cos\theta=\frac{3}{\sqrt{13}}$. Then $f(x)$ can be written as

$$f(x)=\sqrt{13}\left(\frac{2}{\sqrt{13}}\cos x+\frac{3}{\sqrt{13}}\sin x\right)=\sqrt{13}(\sin\theta\cos x+\cos\theta\sin x)=\sqrt{13}\sin(x+\theta)$$

While θ is a constant, x takes all real values and so $\sin(x+\theta)$ will take all values between 1 and -1. Therefore, the smallest value of $f(x)$ is $-\sqrt{13}$ and the largest value of $f(x)$ is $\sqrt{13}$.

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