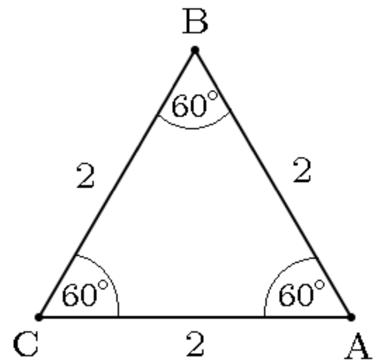
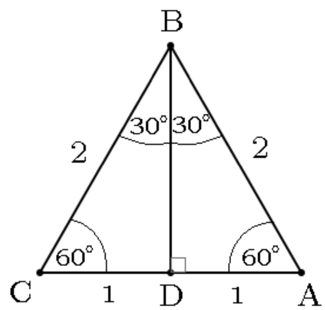
The exact value of trigonometric functions of most angles can not be determined by elementary techniques. At this point, we are to imagine mathematicians drawing right triangles and measuring sides to obtain approximate values. There are a few angles, however, that are exception to this; we can compute the exact values of trigonometric functions. Certain symmetries and the Pythagorean theorem enables us to do that, in case of $30°$, $45°$, and $60°$.

Trigonometric Function Values of $30°$ and $60°$

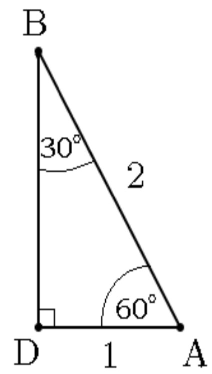
We start by drawing an equilateral triangle with sides 2 units long. All three angles of this triangle measure $60°$.



Let D be the midpoint of side AC. We connect points B and D. Because the triangle is isosceles, this line is perpendicular to the base AC and cuts the triangle into two identical parts. Consequently, the two angles created at point B both measure $30°$.



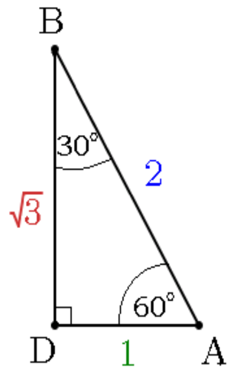
Let us now focus on just one half of the picture, triangle ABD. This triangle has angles $30°$, $60°$, and $90°$. We know two of its sides are 1 and 2 units long. Notice that the hypotenuse is 2 units long.



We can easily compute the missing side using the Pythagorean theorem. If side BD is denoted by x, then

$$\begin{aligned} x^2 + 1^2 &= 2^2 \\ x^2 + 1 &= 4 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \quad \Longrightarrow \quad x = \sqrt{3} \end{aligned}$$

Now that we have the exact value of all three sides of the triangle, we can compute all trigonometric function values for 30° and 60° using this triangle.



Trigonometric Function Values Of 30°:

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \csc 30^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{1} = 2$$

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Trigonometric Function Values Of 60°:

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \csc 60^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sec 60^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{1} = 2$$

$$\tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot 60^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\sqrt{3}}$$

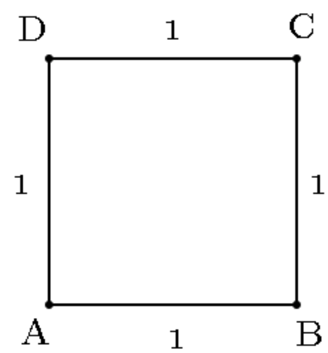
Note: Consider the exact value of $\tan 30^\circ$. The ratio of the sides is $\frac{1}{\sqrt{3}}$. Sometimes this number is rationalized and is presented as $\frac{\sqrt{3}}{3}$. Here is the computation:

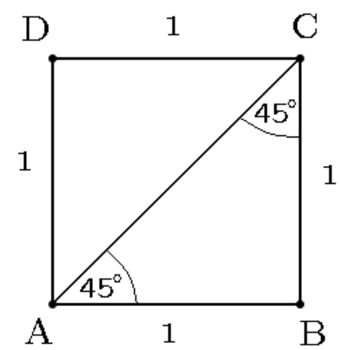
$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot 1 = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Similarly, $\csc 60° = \dfrac{2}{\sqrt{3}}$ can be rationalized and presented as $\dfrac{2\sqrt{3}}{3}$. While the rationalized form is considered 'simplified', both forms have their own adventages, and it is a useful skill to know which form is better for us in different situations. For example, if we add several fractions of different denominators and $\tan 30°$ is one of them, the rationalized form is much better. On the other hand, if we need to square $\tan 30°$, it is easier to work with the other form: $(\tan 30°)^2 = \left(\dfrac{1}{\sqrt{3}}\right)^2 = \dfrac{1}{3}$.

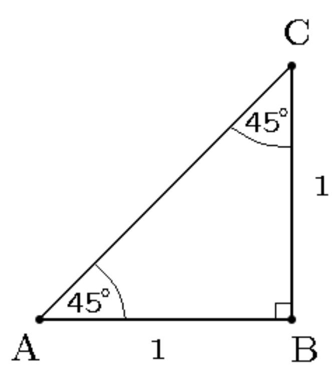
Trigonometric Function Values of 45°

We start by drawing a square with sides 1 units long. We then draw the diagonal AC. This line cuts the square into two identical, isosceles right triangles. Consequently, the angles created at points A and C both measure 45°.





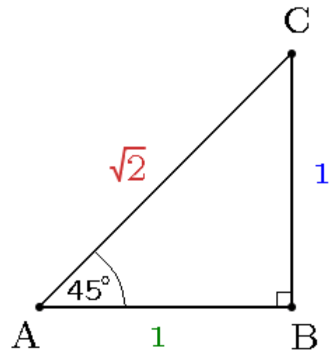
Consider now triangle ABC.



We compute the hypotenuse AC by the Pythagorean theorem: if we denote AC by x, we have that

$$\begin{aligned} 1^2 + 1^2 &= x^2 \\ 2 &= x^2 \\ x &= \pm\sqrt{2} \quad \Longrightarrow \quad x = \sqrt{2} \end{aligned}$$

To avoid confusion, we mark only one of the 45° angle on the picture below. Now that we know all three sides of the triangle, we can compute all trigonometric values of 45°.



Trigonometric Function Values Of 45°:

$$\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \qquad \cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

Note: $\sin 45^\circ = \frac{1}{\sqrt{2}}$ can be rationalized: $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Again, while the rationalized form is considered 'simplified', both forms have their own adventages.

Note: There are some additional angles with algebraically approachable trigonometric function values. We can also compute the exact values for the trigonometric functions of 18° and 72°. These computations use the Pythagorean theorem and similar triangles and are very interesting.

