Theorem: For all angles α,

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \quad \text{and} \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Proof. Let us start with the double-angle formula for cosine. We will use the form that only involves cosine and solve for $\cos x$.

$$\begin{aligned}
\cos 2x &= 2\cos^2 x - 1 && \text{add } 1\\
\cos 2x + 1 &= 2\cos^2 x && \text{divide by } 2\\
\frac{\cos 2x + 1}{2} &= \cos^2 x\\
\pm\sqrt{\frac{\cos 2x + 1}{2}} &= \cos x
\end{aligned}$$

Now we will subsititute $2x = \alpha$ and divide both sides by 2, we get $x = \frac{\alpha}{2}$.

$$\cos x = \pm\sqrt{\frac{\cos 2x + 1}{2}} \quad \text{becomes} \quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{\cos\alpha + 1}{2}}$$

We prove the half-angle formula for sine similary. We start with the double-angle formula for cosine. We will use the form that only involves sine and solve for $\sin x$.

$$\begin{aligned}
\cos 2x &= 1 - 2\sin^2 x && \text{add } 2\sin^2 x\\
\cos 2x + 2\sin^2 x &= 1 && \text{subtract } 2\cos x\\
2\sin^2 x &= 1 - \cos 2x && \text{divide by } 2\\
\sin^2 x &= \frac{1-\cos 2x}{2}\\
\sin x &= \pm\sqrt{\frac{1-\cos 2x}{2}}
\end{aligned}$$

Now we subsititute $2x = \alpha$ and divide both sides by 2, we get $x = \frac{\alpha}{2}$.

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}} \quad \text{becomes} \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

What about the $\pm$? Are both solutions correct? The answer is yes, they might both be correct and we need to know the half angle's location to sort out which one is the correct value for the half-angle.

Consider that we do not know what the angle α is, only its sine and cosine. If we know the sine and cosine of an angle, we still do not know its value, only up to co-terminal angles, that is, any multiple of $360°$ added. And if two angles differ by $360°$, their halves will differ by $180°$. If those half angles differ by $180°$, then their sines and cosines will be opposites.

Consider for example $120°$ and $480°$. Since these angles are co-terminal, they have the same sine and cosine.

$$\begin{aligned}
\sin 120° &= \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 120° = -\frac{1}{2}\\
\sin 480° &= \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 480° = -\frac{1}{2}
\end{aligned}$$

Let us apply the half-angle formula for both angles

$$\sin 60^\circ = \sin\frac{120^\circ}{2} = \pm\sqrt{\frac{1-\cos 120^\circ}{2}} = \pm\sqrt{\frac{1-\left(-\frac{1}{2}\right)}{2}} = \pm\sqrt{\frac{\frac{3}{2}}{2}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

Since 60° is in the first quadrant, the answer is $\dfrac{\sqrt{3}}{2}$.

$$\sin 240^\circ = \sin\frac{480^\circ}{2} = \pm\sqrt{\frac{1-\cos 480^\circ}{2}} = \pm\sqrt{\frac{1-\left(-\frac{1}{2}\right)}{2}} = \pm\sqrt{\frac{\frac{3}{2}}{2}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

Since 240° is in the third quadrant, the answer is $-\dfrac{\sqrt{3}}{2}$.

When using the half-angle formulas for sine and cosine, we must be aware that the formula does not give us the sign of the half-angle. For the sign, we must figure out in which quadrant the half-angle lies.

Now for the half-angle formula for tangent:

> *Theorem:* For all angles $\alpha \neq k\pi$ where $k \in \mathbb{Z}$,
>
> $$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{1-\cos\alpha}{2}\cdot\frac{2}{1+\cos\alpha}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}\cdot\frac{1-\cos\alpha}{1-\cos\alpha}}$$

$$= \pm\sqrt{\frac{(1-\cos\alpha)^2}{(1+\cos\alpha)(1-\cos\alpha)}} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{(1-\cos^2\alpha)}} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}} = \pm\frac{1-\cos\alpha}{\sin\alpha}$$

For the other form of the formula

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha} = \frac{1-\cos^2\alpha}{\sin\alpha(1+\cos\alpha)} = \frac{\sin^2\alpha}{\sin\alpha(1+\cos\alpha)} = \frac{\sin\alpha}{1+\cos\alpha}$$

And so we have that

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

Where did the $\pm$ go? We only proved the results up to a minus sign. The following alternative proof shows us that there is no ambiguity here.

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\cdot 1 = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\frac{2\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{\sin\alpha}{1+\cos\alpha}$$

The numerator is $\sin\alpha$ by the double-angle formula for sine. The denominator is $1+\cos\alpha$ because of the double-angle formula:

$$\begin{aligned} \cos 2x &= 2\cos^2 x - 1 && \text{Let } 2x=\alpha \quad \text{then} \quad x=\frac{\alpha}{2} \\ \cos\alpha &= 2\cos^2\frac{\alpha}{2} - 1 && \text{add } 1 \\ \cos\alpha + 1 &= 2\cos^2\frac{\alpha}{2} \end{aligned}$$

And so the half-angle formula for tangent has no ambiguity about the sign like the half-angle formulas for since and cosine.

Sample Problems

1. Compute the exact value of each of the following.

   a) $\sin 22.5°$ b) $\cos 22.5°$ c) $\tan 22.5°$

2. Compute the exact value of $\cos x$ if we know that $\cos 2x = -\dfrac{1}{3}$.
3. Compute the exact value of $\sin\dfrac{\beta}{2}$ if we know that $\tan\beta = -2$.

Practice Problems

1. Compute the exact value of $\sin 105°$ using a half-angle formula.
2. Compute the exact value of each of the following.

   a) $\sin 15°$ c) $\sin 7.5°$ e) $\tan 7.5°$
   b) $\cos 15°$ d) $\cos 7.5°$

3. Compute the exact value of $\sin x$ and $\cos x$ if x is in the second quadrant and $\cos 2x = \dfrac{7}{25}$.
4. Compute the exact value of $\cos\dfrac{x}{2}$ and $\tan\dfrac{x}{2}$ if we know that $\cos x = \dfrac{2}{3}$.
5. Compute the exact value of $\sin\alpha$ if $\sin 2\alpha = \dfrac{4}{5}$.

Sample Problems - Answers

1.) a) $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ b) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ c) $\sqrt{2}-1$ 2.) $\dfrac{\sqrt{3}}{3}$ 3.) $\pm\sqrt{\dfrac{5\pm\sqrt{5}}{10}}$

Practice Problems - Answers

1. $\dfrac{\sqrt{2+\sqrt{3}}}{2}$

2. a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ c) $\dfrac{1}{2}\sqrt{\dfrac{4-\sqrt{6}-\sqrt{2}}{2}}$ d) $\dfrac{1}{2}\sqrt{\dfrac{4+\sqrt{6}+\sqrt{2}}{2}}$

   e) $\dfrac{\sqrt{6}-\sqrt{2}}{4+\sqrt{6}+\sqrt{2}} = -2+\sqrt{6}+\sqrt{2}-\sqrt{3}$

3. $\sin x = \dfrac{3}{5}$ $\cos x = -\dfrac{4}{5}$

4. $\cos\dfrac{x}{2} = \pm\dfrac{\sqrt{30}}{6}$ and $\tan\dfrac{x}{2} = \pm\dfrac{\sqrt{5}}{5}$

5. $\pm\dfrac{2\sqrt{5}}{5}, \pm\dfrac{1\sqrt{5}}{5}$

Sample Problems - Solutions

1. Compute the exact value of each of the following.

   a) $\sin 22.5°$

   Solution: $22.5°$ is half of $45°$. So we will apply the appropriate half-angle formula.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

   Let $\alpha = 45°$. Then the formula becomes

$$\begin{aligned}\sin 22.5° &= \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{1}{2}\left(\frac{2}{2}-\frac{\sqrt{2}}{2}\right)} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \pm\frac{\sqrt{2-\sqrt{2}}}{\sqrt{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

   Since $22.5°$ is an ange in the first quadrant, we know that its sine is positive. So the correct answer is $\boxed{\sin 22.5° = \dfrac{\sqrt{2-\sqrt{2}}}{2}}$

b) $\cos 22.5^\circ$

Solution: 22.5° is half of 45°. So we will apply the appropriate half-angle formula.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Let $\alpha = 45^\circ$. Then the formula becomes

$$\begin{aligned}\cos 22.5^\circ &= \pm\sqrt{\frac{1+\cos 45^\circ}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{1}{2}\left(\frac{2}{2}+\frac{\sqrt{2}}{2}\right)} = \pm\sqrt{\frac{2+\sqrt{2}}{4}} \\ &= \pm\frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}} = \pm\frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

Since 22.5° is an ange in the first quadrant, we know that its cosine is positive. So the correct answer is $\boxed{\cos 22.5^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}}$

c) $\tan 22.5^\circ$

Solution: 22.5° is half of 45°. So we will apply the appropriate half-angle formula.

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

Let $\alpha = 45^\circ$. Then the formula becomes

$$\begin{aligned}\tan 22.5^\circ &= \frac{\sin 45^\circ}{1+\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{2+\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}\cdot\frac{2}{2+\sqrt{2}} = \frac{\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}} \\ &= \frac{\sqrt{2}\left(2-\sqrt{2}\right)}{4-2} = \frac{\left(2\sqrt{2}-2\right)}{2} = \frac{2\left(\sqrt{2}-1\right)}{2} = \sqrt{2}-1\end{aligned}$$

So the answer is $\boxed{\tan 22.5^\circ = \sqrt{2}-1}$

2. Compute the exact value of $\cos x$ if we know that $\cos 2x = -\frac{1}{3}$.

Solution: We will apply the appropriate half-angle formula.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

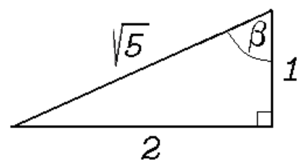
Let $\alpha = 2x$. Then the formula becomes

$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}} = \pm\sqrt{\frac{1+\left(-\frac{1}{3}\right)}{2}} = \pm\sqrt{\frac{\frac{2}{3}}{2}} = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$

Since we don't know which quadrant this angle lies, the answer could be positive or negative. So the correct answer is $\boxed{\cos x = \frac{\sqrt{3}}{3}}$

3. Compute the exact value of $\sin\frac{\beta}{2}$ if we know that $\tan\beta = -2$.

Solution: We first compute $\cos\beta$. To find the numerical value, we draw a right triangle in which $\tan\beta = 2$.



We compute the hypotenuse using the Pythagorean theorem and then we have $\cos\beta$ up to its sign: $\cos\beta = \pm\frac{1}{\sqrt{5}} = \pm\frac{\sqrt{5}}{5}$. Since we have no additional information on β, both answers could be correct. Because of this, we will have four possible values for $\sin\frac{\beta}{2}$:

$$\sin\frac{\beta}{2} = \pm\sqrt{\frac{1-\cos\beta}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{5}}{5}}{2}} = \pm\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}} = \pm\sqrt{\frac{5-\sqrt{5}}{10}} \quad \text{or}$$

$$\sin\frac{\beta}{2} = \pm\sqrt{\frac{1-\cos\beta}{2}} = \pm\sqrt{\frac{1-\left(-\frac{\sqrt{5}}{5}\right)}{2}} = \pm\sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}} = \pm\sqrt{\frac{5+\sqrt{5}}{10}}$$

so the correct answer is $\boxed{\sin\frac{\beta}{2} = \pm\sqrt{\frac{5\pm\sqrt{5}}{10}}}$.

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