Sample Problems

1. Suppose that θ is an acute angle, i.e. $0 < \theta < 90°$. Compute the exact value of $\cos\theta$ and $\tan\theta$, given that $\sin\theta = \frac{2}{5}$.
2. Suppose that β is an acute angle, i.e. $0 < \beta < 90°$. Compute the exact value of all trigonometric function values of β, given that $\tan\beta = 3$.
3. Suppose that α is an acute angle, i.e. $0 < \alpha < 90°$. Compute the exact value of all trigonometric values of α, given that $\sin\alpha = x$.
4. Suppose that θ is an acute angle, i.e. $0 < \theta < 90°$. Compute the exact value of all trigonometric values of θ, given that $\tan\theta = x$.

Practice Problems

1. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cos\alpha = \frac{3}{7}$. Rationalize the denominator in the answer.
2. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\csc\beta = 4$. Rationalize the denominator in the answer.
3. Compute the exact value of all trigonometric functions of γ if γ is an acute angle with $\sec\gamma = \frac{2}{3}$. Rationalize the denominator in the answer.
4. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cot\alpha = \frac{3}{7}$. Rationalize the denominator in the answer.
5. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\sin\beta = x$.
6. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\tan\alpha = x$.
7. Compute the exact value of all trigonometric functions of θ if θ is an acute angle with $\sec\theta = \frac{5}{2}$.

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Answers - Sample Problems

1. $\cos\theta = \dfrac{\sqrt{21}}{5}$ and $\tan\theta = \dfrac{2}{\sqrt{21}} = \dfrac{2\sqrt{21}}{21}$

2. $\sin\beta = \dfrac{3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{10}$, $\cos\beta = \dfrac{1}{\sqrt{10}} = \dfrac{\sqrt{10}}{10}$, and $\tan\beta = 3$, $\csc\beta = \dfrac{\sqrt{10}}{3}$, $\sec\beta = \dfrac{\sqrt{10}}{1}$, and $\cot\beta = \dfrac{1}{3}$

3. $\sin\alpha = x$, $\cos\alpha = \sqrt{1-x^2}$, $\tan\alpha = \dfrac{x}{\sqrt{1-x^2}}$, $\csc\alpha = \dfrac{1}{x}$, $\sec\alpha = \dfrac{1}{\sqrt{1-x^2}}$, and $\cot\alpha = \dfrac{\sqrt{1-x^2}}{x}$

4. $\sin\theta = \dfrac{x}{\sqrt{x^2+1}}$, $\cos\theta = \dfrac{1}{\sqrt{x^2+1}}$, $\tan\theta = x$, $\csc\theta = \dfrac{\sqrt{x^2+1}}{x}$, $\sec\theta = \sqrt{x^2+1}$, $\cot\theta = \dfrac{1}{x}$

Practice Problems - Answers

1. $\sin\alpha = \dfrac{2\sqrt{10}}{7}$, $\cos\alpha = \dfrac{3}{7}$, $\tan\alpha = \dfrac{2\sqrt{10}}{3}$, $\csc\alpha = \dfrac{7\sqrt{10}}{20}$, $\sec\alpha = \dfrac{7}{3}$, $\cot\alpha = \dfrac{3\sqrt{10}}{20}$

2. $\sin\beta = \dfrac{1}{4}$, $\cos\beta = \dfrac{\sqrt{15}}{4}$, $\tan\beta = \dfrac{\sqrt{15}}{15}$, $\csc\beta = 4$, $\sec\beta = \dfrac{4\sqrt{15}}{15}$, $\cot\beta = \sqrt{15}$

3. This is impossible, $\sec\gamma$ must have a value greater than one. There is no angle with $\sec\gamma = \dfrac{2}{3}$.

4. $\sin\alpha = \dfrac{7\sqrt{58}}{58}$, $\cos\alpha = \dfrac{3\sqrt{58}}{58}$, $\tan\alpha = \dfrac{7}{3}$, $\csc\alpha = \dfrac{\sqrt{58}}{7}$, $\sec\alpha = \dfrac{\sqrt{58}}{3}$, $\cot\alpha = \dfrac{3}{7}$

5. $\sin\beta = x$, $\cos\beta = \sqrt{1-x^2}$, $\tan\beta = \dfrac{x}{\sqrt{1-x^2}}$, $\csc\beta = \dfrac{1}{x}$, $\sec\beta = \dfrac{1}{\sqrt{1-x^2}}$, $\cot\beta = \dfrac{\sqrt{1-x^2}}{x}$

6. $\sin\alpha = \dfrac{x}{\sqrt{x^2+1}}$, $\cos\alpha = \dfrac{1}{\sqrt{x^2+1}}$, $\tan\alpha = x$, $\csc\alpha = \dfrac{\sqrt{x^2+1}}{x}$, $\sec\alpha = \sqrt{x^2+1}$, $\cot\alpha = \dfrac{1}{x}$

7. $\sin\theta = \dfrac{\sqrt{21}}{5}$, $\cos\theta = \dfrac{2}{5}$, $\tan\theta = \dfrac{\sqrt{21}}{2}$, $\csc\theta = \dfrac{5\sqrt{21}}{21}$, $\sec\theta = \dfrac{5}{2}$, $\cot\theta = \dfrac{2\sqrt{21}}{21}$

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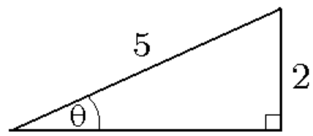
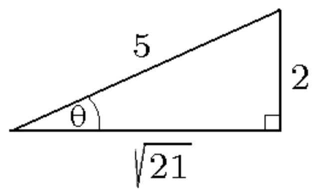
Sample Problems - Solutions

1. Suppose that θ is an acute angle, i.e. $0 < \theta < 90°$. Compute the exact value of $\cos\theta$ and $\tan\theta$, given that $\sin\theta = \frac{2}{5}$.

Solution 1 (Geometrical approach) It is given that $\sin\theta = \frac{2}{5}$. We draw a right triangle where this happens. We draw a hypotenuse of length 5 units and a shorter side of length 2 units and draw in θ so that $\sin\theta = \frac{2}{5}$ is true in the triangle.



We now apply the Pythagorean theorem to find the length of the missing side, $\sqrt{21}$.



We can now easily read all other trigonometric function values of θ from the picture.

$$\cos\theta = \frac{\sqrt{21}}{5} \quad \text{and} \quad \tan\theta = \frac{2}{\sqrt{21}}$$

We finally rationalize the value for $\tan\theta$:

$$\tan\theta = \frac{2}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1-\sin^2\theta} \end{aligned}$$

Since θ is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

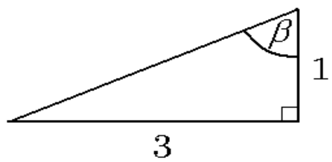
$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{2}{5}\right)^2} = \sqrt{1-\frac{4}{25}} = \sqrt{\frac{25-4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{\sqrt{25}} = \frac{\sqrt{21}}{5}$$

Now we can compute all other trigonometric values of θ

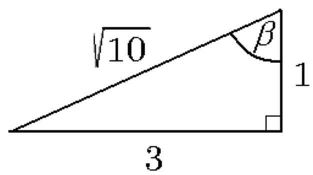
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} = \frac{2}{5} \cdot \frac{5}{\sqrt{21}} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

2. Suppose that β is an acute angle, i.e. $0 < \beta < 90°$. Compute the exact value of all trigonometric function values of β, given that $\tan\beta = 3$.

Solution 1 (Geometrical approach) It is given that $\tan\beta = 3$. We will think of 3 as $\dfrac{3}{1}$ and draw a right triangle where $\tan\beta = \dfrac{3}{1}$ happens. That is a right triangle with shorter sides measuring 1 and 3 units, where the angle opposite the longer side is β and so $\tan\beta = 3$ is true in the triangle.



We now apply the Pythagorean theorem to find the hypotenuse. We obtain $\sqrt{10}$.



We can now use the picture to read all other trigonometric function values of β. We also rationalize all values.

$$\begin{aligned} \sin\beta &= \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}, \quad \cos\beta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}, \quad \text{and} \quad \tan\beta = \frac{3}{1} = 3 \\ \csc\beta &= \frac{\sqrt{10}}{3}, \quad \sec\beta = \frac{\sqrt{10}}{1}, \quad \text{and} \quad \cot\beta = \frac{1}{3} \end{aligned}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos\beta$.

$$\begin{aligned} \sin^2\beta + \cos^2\beta &= 1 \qquad \text{divide by } \cos^2\beta \\ \frac{\sin^2\beta}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\beta} &= \frac{1}{\cos^2\beta} \\ \tan^2\beta + 1 &= \frac{1}{\cos^2\beta} \end{aligned}$$

Recall that $\dfrac{1}{\cos x} = \sec x$. The identity we derived, $\tan^2 x + 1 = \sec^2 x$ will play a very important role in calculus. We will proceed to solve for $\cos\beta$.

$$\begin{aligned} \frac{1}{\cos^2\beta} &= \tan^2\beta + 1 \qquad \text{take reciprocal of both sides} \\ \cos^2\beta &= \frac{1}{\tan^2\beta + 1} \\ \cos\beta &= \pm\sqrt{\frac{1}{\tan^2\beta + 1}} = \pm\frac{1}{\sqrt{\tan^2\beta + 1}} \end{aligned}$$

Since β is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

$$\cos\beta = \sqrt{\frac{1}{\tan^2\beta + 1}} = \sqrt{\frac{1}{3^2 + 1}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

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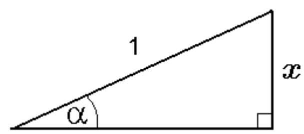
Now we can compute all other trigonometric values of β.

$$\begin{aligned}\tan\beta &= \frac{\sin\beta}{\cos\beta} \qquad \text{multiply by } \cos\beta\\ \tan\beta\cos\beta &= \sin\beta\\ \sin\beta &= 3\cdot\frac{\sqrt{10}}{10} = \frac{3\sqrt{10}}{10}\end{aligned}$$

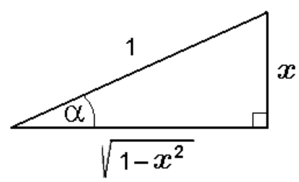
So $\sin\beta = \dfrac{3\sqrt{10}}{10}$, $\cos\beta = \dfrac{\sqrt{10}}{10}$, $\tan\beta = 3$, $\csc\beta = \dfrac{\sqrt{10}}{3}$, $\sec\beta = \sqrt{10}$, and $\cot\beta = \dfrac{1}{3}$

3. Suppose that α is an acute angle, i.e. $0 < \alpha < 90°$. Compute the exact value of all trigonometric values of α, given that $\sin\alpha = x$.

Solution 1 (Geometrical approach) It is given that $\sin\alpha = x$. We can note right now that because x is the sine of an acute angle, x can not take just any value: it has to be between 0 and 1. We will think of x as $\dfrac{x}{1}$ and draw a right triangle where $\sin\alpha = \dfrac{x}{1}$ happens. We draw a right triangle and label the hypotenuse as 1. We label one angle as α and the side opposite as x.



We now apply the Pythagorean theorem to find the missing side. We obtain $\sqrt{1-x^2}$.



We can now use the picture to read all other trigonometric function values of α.

$$\sin\alpha = \frac{x}{1} = x, \quad \cos\alpha = \sqrt{1-x^2}, \quad \tan\alpha = \frac{x}{\sqrt{1-x^2}}, \quad \csc\alpha = \frac{1}{x}, \quad \sec\alpha = \frac{1}{\sqrt{1-x^2}}, \quad \text{and} \quad \cot\alpha = \frac{\sqrt{1-x^2}}{x}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos\alpha$.

$$\begin{aligned}\sin^2\alpha + \cos^2\alpha &= 1\\ \cos^2\alpha &= 1-\sin^2\alpha\\ \cos\alpha &= \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{1-x^2}\end{aligned}$$

Since α is an acute angle, all trigonometric values are positive and so we can discard the negative solution. So $\cos\alpha = \sqrt{1-x^2}$. Now we can compute all other trigonometric values of α.

$$\sin\alpha = x, \quad \cos\alpha = \sqrt{1-x^2}, \quad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{x}{\sqrt{1-x^2}}, \quad \csc\alpha = \frac{1}{x}, \quad \sec\alpha = \frac{1}{\sqrt{1-x^2}}, \quad \cot\alpha = \frac{\sqrt{1-x^2}}{x}$$

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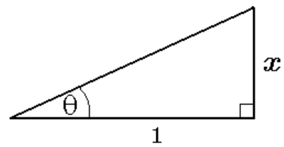
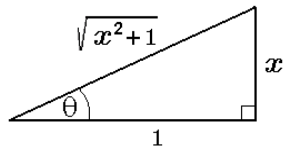
4. Suppose that θ is an acute angle, i.e. $0 < \theta < 90°$. Compute the exact value of all trigonometric values of θ, given that $\tan\theta = x$.

Solution 1 (Geometrical approach) It is given that $\tan\theta = x$. We will think of x as $\frac{x}{1}$ and draw a right triangle where $\tan\theta = \frac{x}{1}$ happens. We draw a right triangle and label the shorter sides as x and 1. We label the angle opposite x as θ.



We now apply the Pythagorean theorem to find the hypotenuse. We obtain $\sqrt{x^2+1}$.



We can now use the picture to read all other trigonometric function values of θ.

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}, \quad \cos\theta = \frac{1}{\sqrt{x^2+1}}, \quad \tan\theta = \frac{x}{1} = x, \quad \csc\theta = \frac{\sqrt{x^2+1}}{x}, \quad \sec\theta = \sqrt{x^2+1}, \quad \text{and} \quad \cot\alpha = \frac{1}{x}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos\theta$.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 && \text{divide by } \cos^2\theta \\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
\tan^2\theta + 1 &= \frac{1}{\cos^2\theta} && \text{take reciprocal of both sides} \\
\frac{1}{\tan^2\theta + 1} &= \cos^2\theta \\
\cos\theta &= \pm\sqrt{\frac{1}{\tan^2\theta+1}} = \pm\frac{1}{\sqrt{\tan^2\theta+1}} = \pm\frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

Since θ is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

Now we can compute all other trigonometric values of θ.

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta} && \text{multiply by } \cos\theta \\
\tan\theta\cos\theta &= \sin\theta \\
\sin\theta &= \tan\theta\cos\theta = x \cdot \frac{1}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}
\end{aligned}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}, \quad \cos\theta = \frac{1}{\sqrt{x^2+1}}, \quad \tan\theta = x, \quad \csc\theta = \frac{\sqrt{x^2+1}}{x}, \quad \sec\theta = \sqrt{x^2+1}, \quad \cot\theta = \frac{1}{x}$$

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