

Sample Problems

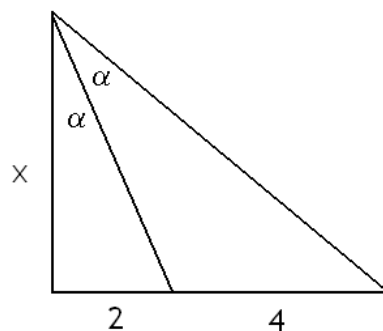
Assume the following identities: For all x, y real numbers,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad \text{and} \quad \cos(x + y) = \cos x \cos y - \sin x \sin y$$

1. Find the formula for $\tan(x + y)$ in terms of $\tan x$ and $\tan y$.
2. Double-angle formulas.
 - a) Find the formula for $\sin 2\alpha$.
 - b) Find the formula for $\cos 2\alpha$.
 - c) Find all three forms of the double-angle formula for $\cos 2\alpha$.
 - d) The double-angle formula for tangent.
 - i) Use the identities for $\sin 2\alpha$ and $\cos 2\alpha$ to derive the formula for $\tan 2\alpha$.
 - ii) Use the identity $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ to derive the formula for $\tan 2\alpha$.
3. Difference formulas.
 - a) Use the unit circle to prove that for all x , $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
 - b) Find and prove the formula for $\sin(x - y)$.
 - c) Find and prove the formula for $\cos(x - y)$.
 - d) The difference formula for tangent.
 - i) Use the sum formula to derive the difference formula for tangent.
 - ii) Use the identities for $\sin(x - y)$ and $\cos(x - y)$ to derive the formula for $\tan(x - y)$.
4. Which of the following is NOT equivalent to $\cos(x + 270^\circ)$?
 A) $\sin x$ B) $\cos(x - 90^\circ)$ C) $\cos(90^\circ - x)$ D) $\sin(360^\circ - x)$ E) $\sin(180^\circ - x)$
5. If simplified, the expression $1 - \sin 2x \tan x$ is equivalent to which one of the following?
 A) $\cos 2x$ B) $\cos^2 x$ C) $\sin x \cos x$ D) $2 \cos x$ E) $\sin^2 x - \cos^2 x$
6. Find the exact value for each of the following expressions.

a) $\cos 75^\circ$	d) $\sin 80^\circ \cos 50^\circ - \cos 80^\circ \sin 50^\circ$	g) $\frac{2 \tan \frac{5\pi}{12}}{1 - \tan^2 \frac{5\pi}{12}}$
b) $\cos 68^\circ \sin 8^\circ - \sin 68^\circ \cos 8^\circ$	e) $\sin 80^\circ \cos 10^\circ + \cos 80^\circ \sin 10^\circ$	h) $\tan 22.5^\circ$
c) $\frac{\tan \frac{2\pi}{15} + \tan \frac{\pi}{5}}{1 - \left(\tan \frac{2\pi}{15}\right) \left(\tan \frac{\pi}{5}\right)}$	f) $\sin 195^\circ$	
7. Solve each of the following equations. Present your answer as exact values, in radians.
 - a) $2 + 3 \sin x = \cos 2x$
 - b) $2 \sin^2 x + \sin 2x = 0$
8. Suppose that $\sin \alpha = -\frac{8}{17}$ and α is not in the fourth quadrant; $\cos \beta = \frac{12}{13}$ and β is not in the first quadrant. Find the exact value for each of the following.
 - a) $\tan(\alpha - \beta)$
 - b) $\cos(\alpha + \beta)$
 - c) $\sin 2\alpha$
9. Prove each of the following identities.
 - a) $\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$
 - b) $4 \sin^4 x = 1 - 2 \cos 2x + \cos^2 2x$
 - c) $\cos 3x = 4 \cos^3 x - 3 \cos x$

10. Find the exact value of $\tan \alpha$ if α is the acute angle formed by the lines $2x - 3y = 5$ and $5x + 3y = 1$.
11. Compute $\tan \theta$ if we know that $\tan 2\theta = \frac{4}{3}$.
12. Let l be the line $y = \frac{3}{4}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the x -axis.
13. Find $\sin \alpha$ if $\cos 2\alpha = \frac{7}{9}$ and α is in the first quadrant.
14. Find $\tan y$ if we know that $\tan x = 3$ and $\tan(x + y) = 33$.
15. Find the exact values of x and α , based on the picture below.



Practice Problems

- Find and prove the formula for $\cot(x + y)$.
- Find the exact value for each of the following expressions.
 - $\sin 15^\circ$
 - $\cos^2 15^\circ - \sin^2 15^\circ$
 - $\cos 255^\circ$
 - $\frac{\tan 78^\circ - \tan 18^\circ}{1 + \tan 78^\circ \tan 18^\circ}$
 - $\cos\left(\frac{25\pi}{12}\right) \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{25\pi}{12}\right) \cos\left(\frac{\pi}{3}\right)$
 - $\sin 22.5^\circ$
 - $\cos 48^\circ \cos 3^\circ + \sin 48^\circ \sin 3^\circ$
- Prove each of the following identities.
 - $\frac{1 - \sin 2x}{\cos 2x} = \frac{1 - \tan x}{1 + \tan x}$
 - $\frac{1 - \tan \frac{1}{2}x}{1 + \tan \frac{1}{2}x} = \frac{1 - \sin x}{\cos x}$
 - $\sin^2 \frac{x}{2} = \frac{\csc x - \cot x}{2 \csc x}$
 - $\sin 3x = 3 \sin x - 4 \sin^3 x$
 - $\tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y}$
 - $\cos 4t = 8 \cos^4 t - 8 \cos^2 t + 1$
 - $\sin\left(x + \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{3}\right) = \sqrt{3} \sin x$
 - $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

4. Suppose that $\sin \alpha = -\frac{24}{25}$ and α is not in the fourth quadrant, and that $\cos \beta = \frac{5}{13}$ and β is not in the first quadrant. Find the **exact value** for each of the following.
- a) $\cos 2\alpha$ c) $\sin(\alpha + \beta)$ e) $\tan 2\alpha$ g) $\tan(\alpha - \beta)$
 b) $\sin 2\beta$ d) $\cos(\beta - \alpha)$ f) $\tan(\alpha + \beta)$ h) $\sin 3\beta$
5. Suppose that $\sin \alpha = \frac{3}{5}$ and α lies in quadrant I, and $\sin \beta = \frac{12}{13}$, and β lies in quadrant II. Find the exact values for each of the following
- a) $\sin 2\alpha$ c) $\sin(\alpha + \beta)$ e) $\tan(\alpha + \beta)$
 b) $\cos 2\alpha$ d) $\cos(\alpha - \beta)$ f) $\sin 3\alpha$
6. Find the exact value of $\tan \alpha$ if we know that $\tan \beta = \frac{1}{3}$ and $\tan(\alpha + \beta) = \frac{1}{2}$.
7. Solve each of the following equations.
- a) $\cos 2x - \sin x = 1$ b) $\cos 2x + 3 \sin x = 2$ c) $\sin x = \sin 2x$
8. Find the exact value of $\cos \alpha$ if $\cos 2\alpha = -\frac{119}{169}$ and α is in the second quadrant.
9. Let m be the straight line determined by the equation $2x - 3y = 8$ and k the line determined by $y = 6x - 15$. Find the exact value of the tangent of the angle formed by these lines.
10. Let l be the line $y = \frac{4}{3}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the x -axis.
11. The figure below consists of three squares. Find the exact value of each of the following.



- a) $\tan \alpha$ b) $\tan \beta$ c) $\tan(\alpha + \beta)$ d) $\tan \gamma$

Sample Problems - Answers

1. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

2. Double-angle formulas.

a) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

b) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

c) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$\cos 2\alpha = 1 - 2 \sin^2 \alpha$

$\cos 2\alpha = 2 \cos^2 \alpha - 1$

d) $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

3. Difference formulas.

a) see solutions

b) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

c) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

d) $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

4. D

5. A

6. a) $\frac{\sqrt{6} - \sqrt{2}}{4}$ b) $-\frac{\sqrt{3}}{2}$ c) $\sqrt{3}$ d) $\frac{1}{2}$ e) 1 f) $\frac{\sqrt{2} - \sqrt{6}}{4}$ g) $-\frac{\sqrt{3}}{3}$ h) $\sqrt{2} - 1$

7. a) $x = -\frac{\pi}{2} + 2k_1\pi, -\frac{\pi}{6} + 2k_2\pi, -\frac{5\pi}{6} + 2k_3\pi$ where $k_1, k_2, k_3 \in \mathbb{Z}$

b) $x = k_1\pi, x = -\frac{\pi}{4} + k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$

8. a) $\frac{171}{140}$ b) $-\frac{220}{221}$ c) $-\frac{240}{289}$

9. see solutions

10. $\frac{1}{3}$

11. $\frac{1}{2}$ or -2

12. $y = \frac{1}{3}x$

13. $\frac{3}{10}$

14. 21

15. $\alpha = 30^\circ, x = 2\sqrt{3}$

Practice Problems - Answers

1. Find and prove the formula for $\cot(x + y)$.

Solution: Use the sum-formulas for $\sin(x + y)$ and $\cos(x + y)$ and then divide both numerator and denominator by $\sin x \sin y$

$$\begin{aligned} \cot(x + y) &= \frac{\cos(x + y)}{\sin(x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} = \frac{\frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}}{\frac{\sin x \cos y + \cos x \sin y}{\sin x \sin y}} \\ &= \frac{\frac{\cos x \cos y}{\sin x \sin y} - \frac{\sin x \sin y}{\sin x \sin y}}{\frac{\sin x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y}} = \frac{\cot x \cot y - 1}{\cot y + \cot x} \end{aligned}$$

$$2. \text{ a) } \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{b) } \frac{\sqrt{3}}{2} \quad \text{c) } \frac{\sqrt{2} - \sqrt{6}}{4} \quad \text{d) } \sqrt{3} \quad \text{e) } \frac{\sqrt{2}}{2} \quad \text{f) } \frac{1}{2}\sqrt{2 - \sqrt{2}} \quad \text{g) } \frac{1}{\sqrt{2}}$$

3. Prove each of the following identities.

$$\text{a) } \frac{1 - \sin 2x}{\cos 2x} = \frac{1 - \tan x}{1 + \tan x}$$

Solution:

$$\begin{aligned} \text{RHS} &= \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} = \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}} = \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{\cos x - \sin x}{\cos x + \sin x} \cdot \frac{\cos x - \sin x}{\cos x - \sin x} = \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} \\ &= \frac{\cos^2 x + \sin^2 x - 2 \sin x \cos x}{\cos 2x} = \text{LHS} \end{aligned}$$

$$\text{b) } \frac{1 - \tan \frac{1}{2}x}{1 + \tan \frac{1}{2}x} = \frac{1 - \sin x}{\cos x}$$

Solution: Let $y = \frac{1}{2}x$ and then $2y = x$. Then we need to prove $\frac{1 - \tan y}{1 + \tan y} = \frac{1 - \sin 2y}{\cos 2y}$ which we have already proven in part a).

$$\text{c) } \sin^2 \frac{x}{2} = \frac{\csc x - \cot x}{2 \csc x}$$

Solution: Write $y = \frac{x}{2}$. Then we need to prove: $\sin^2 y = \frac{\csc 2y - \cot 2y}{2 \csc 2y}$

$$\text{RHS} = \frac{\frac{1}{\sin 2y} - \frac{\cos 2y}{\sin 2y}}{2 \frac{1}{\sin 2y}}$$

We multiply upstairs and downstairs by $\sin 2y$

$$= \frac{1 - \cos 2y}{2} = \frac{1 - (1 - 2 \sin^2 y)}{2} = \frac{2 \sin^2 y}{2} = \text{LHS}$$

$$\text{d) } \sin 3x = 3 \sin x - 4 \sin^3 x$$

Solution:

$$\begin{aligned} \text{LHS} &= \sin x \cos 2x + \cos x \sin 2x = \sin x (1 - 2 \sin^2 x) + 2 \sin x \cos^2 x \\ &= \sin x - 2 \sin^3 x + 2 \sin x (1 - \sin^2 x) = \text{RHS} \end{aligned}$$

$$\text{e) } \tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y}$$

Solution:

$$\text{LHS} = \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = \frac{\sin(x - y)}{\cos x \cos y} = \text{RHS}$$

$$\text{f) } \cos 4t = 8 \cos^4 t - 8 \cos^2 t + 1$$

Solution: Recall that $\cos 2x = 2 \cos^2 x - 1$

$$\begin{aligned} \text{LHS} &= \cos 4t = 2 \cos^2(2t) - 1 = 2(2 \cos^2 t - 1)^2 - 1 = \\ &= 2(4 \cos^4 t - 4 \cos^2 t + 1) - 1 = 8 \cos^4 t - 8 \cos^2 t + 1 = \text{RHS} \end{aligned}$$

$$g) \sin\left(x + \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{3}\right) = \sqrt{3} \sin x$$

$$\begin{aligned} \text{LHS} &= \sin\left(x + \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{3}\right) \\ &= \sin x \cos\left(\frac{\pi}{6}\right) + \cos x \sin\left(\frac{\pi}{6}\right) - \left(\cos x \cos\left(\frac{\pi}{3}\right) - \sin x \sin\left(\frac{\pi}{3}\right)\right) \\ &= \sin x \left(\frac{\sqrt{3}}{2}\right) + \cos x \left(\frac{1}{2}\right) - \cos x \left(\frac{1}{2}\right) + \sin x \left(\frac{\sqrt{3}}{2}\right) \\ &= \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) \sin x = \sqrt{3} \sin x = \text{RHS} \end{aligned}$$

$$h) \frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$$

$$\text{LHS} = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \text{RHS}$$

$$4. \text{ a) } -\frac{527}{625} \quad \text{b) } -\frac{120}{169} \quad \text{c) } -\frac{36}{325} \quad \text{d) } \frac{253}{325} \quad \text{e) } -\frac{336}{527} \quad \text{f) } \frac{36}{323} \quad \text{g) } -\frac{204}{253} \quad \text{h) } \frac{828}{2197}$$

$$5. \text{ a) } \frac{24}{25} \quad \text{b) } \frac{7}{25} \quad \text{c) } \frac{33}{65} \quad \text{d) } \frac{16}{65} \quad \text{e) } -\frac{33}{56} \quad \text{f) } \frac{117}{125}$$

$$6. \frac{1}{7}$$

$$7. \text{ a) } x = k_1\pi, \quad x = -\frac{\pi}{6} + 2k_2\pi, \quad x = -\frac{5\pi}{6} + 2k_3\pi, \quad \text{where } k_1, k_2, k_3 \in \mathbb{Z}$$

$$\text{b) } x = \frac{\pi}{2} + 2k_1\pi, \quad x = \frac{\pi}{6} + 2k_2\pi, \quad x = \frac{5\pi}{6} + 2k_3\pi, \quad \text{where } k_1, k_2, k_3 \in \mathbb{Z}$$

$$\text{c) } x = k_1\pi, \quad x = \pm\frac{\pi}{3} + 2k_2\pi \quad \text{where } k_1, k_2 \in \mathbb{Z}$$

$$8. -\frac{5}{13}$$

$$9. \frac{16}{15}$$

$$10. y = \frac{1}{2}x$$

$$11. \text{ a) } \frac{1}{3} \quad \text{b) } \frac{1}{2} \quad \text{c) } 1 \quad \text{d) } 1$$

Sample Problems - Solutions

Assume the following identities: For all x, y real numbers,

$$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \text{and} \quad \cos(x+y) = \cos x \cos y - \sin x \sin y}$$

1. Find the formula for $\tan(x+y)$ in terms of $\tan x$ and $\tan y$.

Solution:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

We will now divide both numerator and denominator by $\cos x \cos y$

$$\begin{aligned} \tan(x+y) &= \frac{\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}} \\ &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \end{aligned}$$

2. Double-angle formulas.

- a) Find the formula for $\sin 2\alpha$.

Solution: Let $x = y = \alpha$. Then

$$\sin(x+y) = \sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2 \sin \alpha \cos \alpha$$

- b) Find the formula for $\cos 2\alpha$.

Solution: Let $x = y = \alpha$. Then

$$\cos(x+y) = \cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha = \cos^2 \alpha - \sin^2 \alpha$$

- c) Find all three forms of the double-angle formula for $\cos 2\alpha$.

Solution: We have $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$. We can eliminate $\cos \alpha$ using the identity $\cos^2 \alpha + \sin^2 \alpha = 1$; by re-writing $\cos^2 \alpha = 1 - \sin^2 \alpha$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = (1 - \sin^2 \alpha) - \sin^2 \alpha = 1 - 2 \sin^2 \alpha$$

Similarly, we can eliminate $\sin \alpha$ using the identity $\cos^2 \alpha + \sin^2 \alpha = 1$; by re-writing $\sin^2 \alpha = 1 - \cos^2 \alpha$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \cos^2 \alpha - (1 - \cos^2 \alpha) = 2 \cos^2 \alpha - 1$$

- d) The double-angle formula for tangent.

- i) Use the identities for $\sin 2\alpha$ and $\cos 2\alpha$ to derive the formula for $\tan 2\alpha$.

Solution:

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha - \sin^2 \alpha}$$

We will now divide both numerator and denominator by $\cos^2 \alpha$.

$$\tan 2\alpha = \frac{\frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha}}{\frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha}} = \frac{\frac{2 \sin \alpha}{\cos \alpha}}{\frac{\cos^2 \alpha}{\cos^2 \alpha} - \frac{\sin^2 \alpha}{\cos^2 \alpha}} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

- ii) Use the identity $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ to derive the formula for $\tan 2\alpha$.

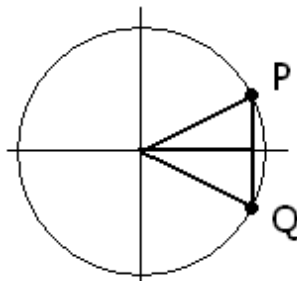
Solution: Let $x = y = \alpha$

$$\tan 2\alpha = \tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

3. Difference formulas.

a) Use the unit circle to prove that for all x , $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

Solution: When we draw an angle x (represented by point P) and its opposite (represented by point Q) we see that the two right triangles created are identical. We see that P and Q have the same first coordinate, thus $\cos(-x) = \cos x$ and that the second coordinates of p and Q are opposites, thus $\sin(-x) = -\sin x$.



b) Find and prove the formula for $\sin(x - y)$.

Solution: We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x - y$ as $x + (-y)$ and apply the sum formula to this sum.

$$\sin(x - y) = \sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y)$$

We know (see part a)) that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$.

$$\sin(x - y) = \sin x \cos y + (-1) \cos x \sin y = \sin x \cos y - \cos x \sin y$$

c) Find and prove the formula for $\cos(x - y)$.

Solution: We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x - y$ as $x + (-y)$ and apply the sum formula to this sum.

$$\cos(x - y) = \cos(x + (-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$

We know (see part a)) that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$.

$$\cos(x - y) = \cos x \cos y - (-1) \sin x \sin y = \cos x \cos y + \sin x \sin y$$

d) The difference formula for tangent.

i) Use the sum formula to derive the difference formula for tangent.

Solution: First, we will prove that $\tan(-x) = -\tan x$ for all real numbers x .

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x - y$ as $x + (-y)$ and apply the sum formula to this sum.

$$\tan(x - y) = \tan(x + (-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x + (-1) \tan y}{1 - (-1) \tan x \tan y} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

ii) Use the identities for $\sin(x - y)$ and $\cos(x - y)$ to derive the formula for $\tan(x - y)$.

Solution:

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

We will now divide both numerator and denominator by $\cos x \cos y$.

$$\begin{aligned}\tan(x - y) &= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} = \frac{\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y}} \\ &= \frac{\frac{\sin x \cos y}{\cos x \cos y} - \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{1 + \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x - \tan y}{1 + \tan x \tan y}\end{aligned}$$

4. Which of the following is NOT equivalent to $\cos(x + 270^\circ)$?

A) $\sin x$ B) $\cos(x - 90^\circ)$ C) $\cos(90^\circ - x)$ D) $\sin(360^\circ - x)$ E) $\sin(180^\circ - x)$

Solution: Let us first simplify $\cos(x + 270^\circ)$, using the sum formula for cosines.

$$\cos(x + 270^\circ) = \cos x \cos 270^\circ - \sin x \sin 270^\circ = \cos x (0) - \sin x (-1) = \sin x$$

and so our expression is equivalent to A. We can similarly simplify all the other expressions using a difference formula.

$$\cos(x - 90^\circ) = \cos x \cos 90^\circ + \sin x \sin 90^\circ = \cos x (0) + \sin x (1) = \sin x$$

so B is also equivalent to $\sin x$. Similarly,

$$\cos(90^\circ - x) = \cos 90^\circ \cos x + \sin 90^\circ \sin x = (0) \cos x + (1) \sin x = \sin x$$

and so C is also equivalent to $\sin x$.

$$\sin(360^\circ - x) = \sin 360^\circ \cos x - \cos 360^\circ \sin x = (0) \cos x - (1) \sin x = -\sin x$$

which is not equivalent to our expression. Finally,

$$\sin(180^\circ - x) = \sin 180^\circ \cos x - \cos 180^\circ \sin x = (0) \cos x - (-1) \sin x = \sin x$$

and so E is also equivalent to $\sin x$.

5. If simplified, the expression $1 - \sin 2x \tan x$ is equivalent to which one of the following?

A) $\cos 2x$ B) $\cos^2 x$ C) $\sin x \cos x$ D) $2 \cos x$ E) $\sin^2 x - \cos^2 x$

Solution:

$$1 - \sin 2x \tan x = 1 - (2 \sin x \cos x) \frac{\sin x}{\cos x} = 1 - 2 \sin^2 x = \cos 2x$$

6. Find the exact value for each of the following expressions.

$$\text{a) } \cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\text{b) } \cos 68^\circ \sin 8^\circ - \sin 68^\circ \cos 8^\circ = \sin(8^\circ - 68^\circ) = \sin(-60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\text{c) } \frac{\tan \frac{2\pi}{15} + \tan \frac{\pi}{5}}{1 - \left(\tan \frac{2\pi}{15}\right) \left(\tan \frac{\pi}{5}\right)} = \tan \left(\frac{2\pi}{15} + \frac{\pi}{5}\right) = \tan \left(\frac{2\pi}{15} + \frac{3\pi}{15}\right) = \tan \left(\frac{5\pi}{15}\right) = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\text{d) } \sin 80^\circ \cos 50^\circ - \cos 80^\circ \sin 50^\circ = \sin(80^\circ - 50^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\text{e) } \sin 80^\circ \cos 10^\circ + \cos 80^\circ \sin 10^\circ =$$

Solution: Since $80^\circ + 10^\circ = 90^\circ$, $\sin 80^\circ = \cos 10^\circ$ and $\cos 80^\circ = \sin 10^\circ$, and so we can re-write the expression as

$$\sin 80^\circ \cos 10^\circ + \cos 80^\circ \sin 10^\circ = \sin 80^\circ \sin 80^\circ + \cos 80^\circ \cos 80^\circ = \sin^2 80^\circ + \cos^2 80^\circ = 1$$

f) $\sin 195^\circ =$

$$\sin(150^\circ + 45^\circ) = \sin 150^\circ \cos 45^\circ + \cos 150^\circ \sin 45^\circ = \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) + \left(-\frac{\sqrt{3}}{2} \right) \left(\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2} - \sqrt{6}}{4}$$

g)
$$\frac{2 \tan \frac{5\pi}{12}}{1 - \tan^2 \frac{5\pi}{12}} = \tan \left(2 \cdot \frac{5\pi}{12} \right) = \tan \frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$$

h) $\tan 22.5^\circ$

Solution: Consider the double-angle formula for tangent. and set $\alpha = 22.5^\circ$.

$$\begin{aligned} \tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} & \alpha &= 22.5^\circ \\ \tan 45^\circ &= \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ} \\ 1 &= \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ} \end{aligned}$$

Now write $a = \tan 22.5^\circ$ and solve the quadratic equation for a .

$$\begin{aligned} 1 &= \frac{2a}{1 - a^2} & \text{multiply by } 1 - a^2 & & 0 &= (a + 1)^2 - 2 \\ 1 - a^2 &= 2a & & & 0 &= (a + 1)^2 - (\sqrt{2})^2 \\ 0 &= a^2 + 2a - 1 & & & 0 &= (a + 1 + \sqrt{2})(a + 1 - \sqrt{2}) \\ 0 &= \underbrace{a^2 + 2a + 1}_{(a+1)^2} - 1 - 1 & & & a_1 &= -1 - \sqrt{2} \quad a_2 = -1 + \sqrt{2} \end{aligned}$$

Since 22.5° is in the first quadrant, its tangent is clearly positive. We can use this to easily rule out a_1 . The answer is $\sqrt{2} - 1$.

7. Solve each of the following equations. Present your answer as exact values, in radians.

a) $2 + 3 \sin x = \cos 2x$

Solution: Substitute $\cos 2x = 1 - 2 \sin^2 x$. Then the equation becomes quadratic in $\sin x$.

$$\begin{aligned} 2 + 3 \sin x &= 1 - 2 \sin^2 x \\ 2 \sin^2 x + 3 \sin x + 1 &= 0 \\ (\sin x + 1)(2 \sin x + 1) &= 0 \\ \sin x + 1 = 0 &\text{ or } 2 \sin x + 1 = 0 \end{aligned}$$

giving us two sets of solutions:

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= -\frac{\pi}{2} + 2k\pi \text{ where } k \text{ is an integer} \end{aligned}$$

$$\begin{aligned} \text{or } 2 \sin x + 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= -\frac{\pi}{6} + 2k\pi \text{ where } k \text{ is an integer and} \\ x &= -\frac{5\pi}{6} + 2k\pi \text{ where } k \text{ is an integer.} \end{aligned}$$

So the set of all solutions is

$$x = -\frac{\pi}{2} + 2k_1\pi \quad \text{and} \quad x = -\frac{\pi}{6} + 2k_2\pi \quad \text{and} \quad -\frac{5\pi}{6} + 2k_3\pi \quad \text{where } k_1, k_2, k_3 \text{ are integers}$$

b) $2 \sin^2 x + \sin 2x = 0 \quad k\pi, \quad x = -\frac{\pi}{4} + k\pi, \text{ where } k \in \mathbb{Z}$

Solution:

$$\begin{aligned} 2 \sin^2 x + \sin 2x &= 0 & \sin 2x &= 2 \sin x \cos x \\ 2 \sin^2 x + 2 \sin x \cos x &= 0 & \text{factor out } \sin x \\ 2 \sin x (\sin x + \cos x) &= 0 \end{aligned}$$

$$\sin x = 0 \quad \text{or} \quad \sin x + \cos x = 0$$

$$\sin x = 0 \implies x = k\pi \quad k \in \mathbb{Z}$$

$$\sin x + \cos x = 0$$

$$\sin x = -\cos x$$

If $\cos x = 0$, then $\sin x = \pm 1$ and so $\sin x = -\cos x$ can not be true. So there is no solution where $\cos x = 0$. We can therefore assume that $\cos x \neq 0$, and then we may divide both sides by $\cos x$.

$$\sin x = -\cos x \quad \text{divide by } \cos x \neq 0$$

$$\frac{\sin x}{\cos x} = -1$$

$$\tan x = -1 \implies x = -\frac{\pi}{4} + k\pi \quad k \in \mathbb{Z}$$

So the set of all solutions is

$$x = k_1\pi \quad \text{and} \quad x = -\frac{\pi}{4} + k_2\pi \quad \text{where } k_1, k_2 \text{ are integers}$$

8. Suppose that $\sin \alpha = -\frac{8}{17}$ and α is not in the fourth quadrant; $\cos \beta = \frac{12}{13}$ and β is not in the first quadrant. Find the exact value for each of the following.

Solution: The conditions given place α into the third quadrant and β into the fourth quadrant. These will determine the signs of the other trigonometric functions.

Since α is in the third quadrant, $\cos \alpha$ is negative. For the rest,

$$\cos \alpha = -\sqrt{1 - \sin^2 \alpha} = -\sqrt{1 - \left(-\frac{8}{17}\right)^2} = -\frac{15}{17}$$

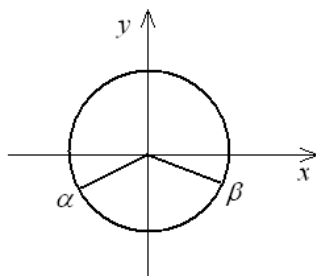
Similarly, since β is in the fourth quadrant, $\sin \beta$ is negative. For the rest,

$$\sin \beta = -\sqrt{1 - \cos^2 \beta} = -\sqrt{1 - \left(\frac{12}{13}\right)^2} = -\frac{5}{13}$$

Now we have all we need:

$$\begin{aligned} \sin \alpha &= -\frac{8}{17} & \cos \alpha &= -\frac{15}{17} & \tan \alpha &= \frac{8}{15} \\ \sin \beta &= -\frac{5}{13} & \cos \beta &= \frac{12}{13} & \tan \beta &= -\frac{5}{12} \end{aligned}$$

Using the calculator, we can also come up with approximate values for α and β . These will give us a practical way to check our solution.



$$\alpha \cong 208.072^\circ \quad \beta \cong -22.6199^\circ$$

$$\text{a) } \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\frac{8}{15} - \left(-\frac{5}{12}\right)}{1 + \frac{8}{15} \left(-\frac{5}{12}\right)} = \frac{\frac{8(4) + 5(5)}{60}}{1 - \frac{2}{9}} = \frac{\frac{57}{60}}{\frac{7}{9}} = \frac{19}{20} \left(\frac{9}{7}\right) = \frac{171}{140}$$

We can use the calculator to check our result by entering the fraction $\frac{171}{140} = 1.22143$ and $\tan(208.072^\circ - (-22.6199^\circ)) = 1.22143$ and compare the decimals.

$$\text{b) } \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{-15}{17} \left(\frac{12}{13}\right) - \frac{-8}{17} \left(\frac{-5}{13}\right) = -\frac{220}{221}$$

$$\text{c) } \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \left(-\frac{8}{17}\right) \left(\frac{15}{17}\right) = -\frac{240}{289}$$

9. Prove each of the following identities.

$$\text{a) } \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$$

$$\begin{aligned} \text{RHS} &= \frac{\cot^2 x - 1}{2 \cot x} = \frac{\frac{\cos^2 x}{\sin^2 x} - 1}{2 \frac{\cos x}{\sin x}} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}{2 \frac{\cos x}{\sin x}} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}{\frac{2 \cos x}{\sin x}} = \\ &= \frac{\cos^2 x - \sin^2 x}{\sin^2 x} \cdot \frac{\sin x}{2 \cos x} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \text{LHS} \end{aligned}$$

$$\text{b) } 4 \sin^4 x = 1 - 2 \cos 2x + \cos^2 2x$$

$$\begin{aligned} \text{RHS} &= 1 - 2 \cos 2x + \cos^2 2x = 1 - 2(1 - 2 \sin^2 x) + (1 - 2 \sin^2 x)^2 = \\ &= 1 - 2 + 4 \sin^2 x + 1 - 4 \sin^2 x + 4 \sin^4 x = 4 \sin^4 x = \text{LHS} \end{aligned}$$

$$\text{c) } \cos 3x = 4 \cos^3 x - 3 \cos x$$

$$\begin{aligned} \text{LHS} &= \cos 3x = \cos(x + 2x) = \cos x \cos 2x - \sin x \sin 2x = \\ &= \cos x (2 \cos^2 x - 1) - \sin x (2 \sin x \cos x) = \\ &= 2 \cos^3 x - \cos x - 2 \sin^2 x \cos x = 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x = \\ &= 2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x) = 2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x = \\ &= 4 \cos^3 x - 3 \cos x = \text{RHS} \end{aligned}$$

10. Find the exact value of $\tan \alpha$ if α is the acute angle formed by the lines $2x - 3y = 5$ and $5x + 3y = 1$.

Solution: Find the slopes of the lines first. We do this by solving the equations for y and then read the coefficients of x .

$$\begin{array}{rcl} 2x - 3y & = & 5 \\ 2x - 5 & = & 3y \\ y & = & \frac{2}{3}x - \frac{5}{3} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 5x + 3y & = & 1 \\ 3y & = & -5x + 1 \\ y & = & -\frac{5}{3}x + \frac{1}{3} \end{array}$$

So the slopes are $m_1 = \tan \alpha = \frac{2}{3}$ and $m_2 = \tan \beta = -\frac{5}{3}$. We are now looking for $\tan(\beta - \alpha)$.

$$\tan(\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha} = \frac{-\frac{5}{3} - \frac{2}{3}}{1 + \left(-\frac{5}{3}\right)\frac{2}{3}} = \frac{-\frac{7}{3}}{1 - \frac{10}{9}} = \frac{-\frac{7}{3}}{-\frac{1}{9}} = -\frac{7}{3}(-9) = 21$$

Note: if we accidentally compute $\tan(\alpha - \beta)$, which is the tangent of the obtuse angle, the result would be -21 , since the tangents of supplemental angles are opposites of each other. Then we would just have to conclude from that that the answer is 21.

11. Compute $\tan \theta$ if we know that $\tan 2\theta = \frac{4}{3}$.

Solution: Consider the double-angle formula for tangent.

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

If we know $\tan 2\theta$, then we can solve for $\tan \theta$. We will introduce a new variable $a = \tan \theta$. then $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ becomes

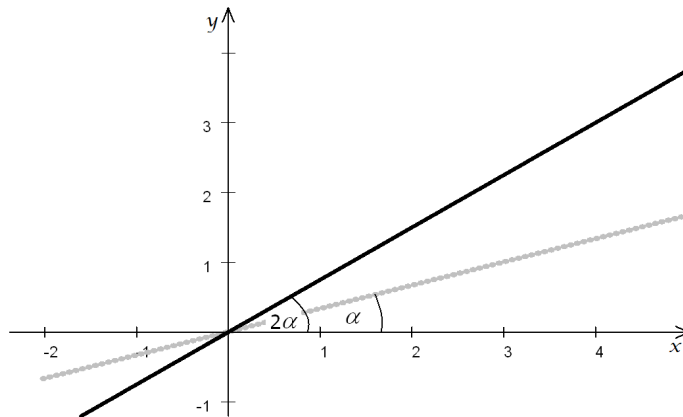
$$\begin{aligned} \frac{4}{3} &= \frac{2a}{1 - a^2} && \text{multiply by } 3(1 - a^2) \\ 4(1 - a^2) &= 6a && \text{divide by 2} \\ 2(1 - a^2) &= 3a \\ 2 - 2a^2 &= 3a \\ 0 &= 2a^2 + 3a - 2 \\ 0 &= (2a - 1)(a + 2) \end{aligned}$$

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_2 = -2$$

Both solutions are correct. (Looking at the tangents $\frac{1}{2}$ and -2 as slopes we see that these two are perpendicular, and so the possible angles for θ differ by 90° . After doubling, these values will differ by 180° and so they will have the same tangents. So, both are correct.)

12. Let l be the line $y = \frac{3}{4}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the x -axis.

Solution: We sketch the problem first. Let α be exactly half of the angle formed between the line $y = \frac{3}{4}x$ and the positive part of the x -axis. Since both α and 2α are clearly acute angles, all trigonometric function values will be positive.



We need to find $\tan \alpha$ if $\tan 2\alpha = \frac{3}{4}$. Let us introduce the new variable $a = \tan \alpha$ and solve for a in the double angle formula for tangents.

$$\begin{aligned}\tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\ \frac{3}{4} &= \frac{2a}{1 - a^2} \quad \text{multiply by } 4(1 - a^2) \\ 3(1 - a^2) &= 4(2a) \\ 3 - 3a^2 &= 8a \\ 0 &= 3a^2 + 8a - 3 \\ 0 &= (3a - 1)(a + 3) \quad \implies a_1 = \frac{1}{3} \quad a_2 = -3\end{aligned}$$

Since $a = \tan \alpha$ must be positive, we rule out -3 and so the answer is $\frac{1}{3}$. This is the slope of the line we are looking for. Since this line passes through the origin, its equation is $y = \frac{1}{3}x$.

13. Find $\sin \alpha$ if $\cos 2\alpha = \frac{7}{9}$ and α is in the first quadrant.

Solution: Solve the equation $\cos 2\alpha = 1 - 2\sin^2 \alpha$ for $\sin \alpha$. Since α is in the first quadrant, $\sin \alpha$ is positive.

$$\begin{aligned}\cos 2\alpha &= 1 - 2\sin^2 \alpha \\ 2\sin^2 \alpha &= 1 - \cos 2\alpha \\ \sin^2 \alpha &= \frac{1 - \cos 2\alpha}{2} \\ \sin \alpha &= \pm \sqrt{\frac{1 - \cos 2\alpha}{2}} = \pm \sqrt{\frac{1 - \frac{7}{9}}{2}} = \pm \frac{1}{3}\end{aligned}$$

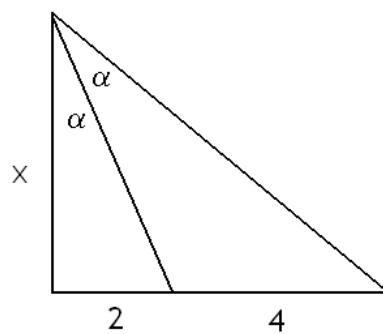
Since $\sin \alpha$ is positive, the answer is $\frac{1}{3}$.

14. Find $\tan y$ if we know that $\tan x = 3$ and $\tan(x + y) = 33$.

Solution: $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. We substitute $\tan x = 3$ and solve for $\tan y$.

$$\begin{aligned}\tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ 33 &= \frac{3 + \tan y}{1 - 3 \tan y} \\ 33(1 - 3 \tan y) &= 3 + \tan y \\ 33 - 99 \tan y &= 3 + \tan y \\ 30 &= 100 \tan y \\ \frac{3}{10} &= \tan y\end{aligned}$$

15. Find the exact values of x and α , based on the picture below.



Solution: $\tan \alpha = \frac{2}{x}$ and $\tan 2\alpha = \frac{6}{x}$.

$$\begin{aligned}\tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\ \frac{6}{x} &= \frac{2 \left(\frac{2}{x}\right)}{1 - \left(\frac{2}{x}\right)^2} && \text{solve for } x \\ \frac{6}{x} &= \frac{\frac{4}{x}}{1 - \frac{4}{x^2}} && \text{divide by 2} \\ \frac{3}{x} &= \frac{2x}{x^2 - 4} && \text{multiply by } x(x^2 - 4) \\ 3(x^2 - 4) &= 2x^2 \\ 3x^2 - 12 &= 2x^2 \\ x^2 &= 12 \implies x = \pm\sqrt{12}\end{aligned}$$

We rule out the negative root since x is a distance. Then $\tan \alpha = \frac{2}{x} = \frac{2}{\sqrt{12}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ thus $\alpha = 30^\circ$.

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