The following trigonometric identities use the sum, difference, and double-angle formulas for sine, cosine, and tangent.

Sample Problems

1. (Co-function identities) Prove each of the following identites using the difference formulas for sine and cosine.

   a) $\sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$ b) $\cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$

2. Prove each of the following identities.

   a) $\dfrac{2}{\sin 2x} = \tan x + \cot x$

   b) $\cos 2x = \dfrac{\cot x - \tan x}{\cot x + \tan x}$

   c) $\dfrac{1 + \cos 2x}{1 - \cos 2x} = \cot^2 x$

   d) $\sin x = \dfrac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

   e) $\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2 = 1 + \sin x$

   f) $\dfrac{\tan 2x}{\tan x} = 1 + \dfrac{1}{\cos 2x}$

   g) $\dfrac{2}{1 - \tan^2 x} = \dfrac{1 + \cos 2x}{\cos 2x}$

   h) $\dfrac{\tan\left(\frac{\pi}{4} - x\right)}{\tan\left(\frac{\pi}{4} + x\right)} = \dfrac{1 - \sin 2x}{1 + \sin 2x}$

   i) $1 + \tan \alpha \tan \beta = \dfrac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta}$

   j) $\sin 35° + \sin 25° = \cos 5°$

   k) $\cos 12° - \cos 48° = \sin 18°$

3. Compute the exact value of $\dfrac{1 + \tan 15°}{1 - \tan 15°}$.

4. Compute the exact value of $\tan \alpha$ if $\sin(\alpha + 60°) = \cos \alpha$

5. Prove that if $\sin(x + y) = 3 \sin(x - y)$, then $\tan x = 2 \tan y$.

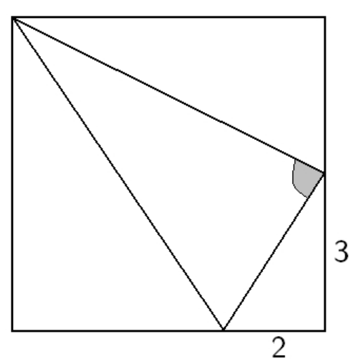
6. Given that $\dfrac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \dfrac{3}{5}$ and $\tan \alpha = \dfrac{1}{2}$, find the exact value of $\tan \beta$.

7. Given that $\tan \alpha$ and $\tan \beta$ are the solutions of the equation $x^2 + 6x + 7 = 0$, show that $\sin(\alpha + \beta) = \cos(\alpha + \beta)$.

8. In triangle ABC, $2 \cos \beta \sin \gamma = \sin \alpha$. Prove that the triangle is isosceles.

9. Given that $\alpha + \beta + \gamma = 90°$, where α, β, and γ are acute angles. Prove that
   $\cot \alpha \cot \beta \cot \gamma = \cot \alpha + \cot \beta + \cot \gamma$.

10. Find the exact value of the tangent of the angle shaded on the picture. The quadrilateral is a square with sides 6 units long.



11. Find the exact value of $\sin 2x$ if we know that $1 + \tan x = \dfrac{35}{12} \sin x$.

Sample Problems - Solutions

1. (Co-function identities) Prove each of the following identites using the difference formulas for sine and cosine.

   a) $\sin\alpha = \cos\left(\frac{\pi}{2}-\alpha\right)$

   Proof:
$$\text{RHS} = \cos\left(\frac{\pi}{2}-\alpha\right) = \cos\left(\frac{\pi}{2}\right)\cos\alpha + \sin\left(\frac{\pi}{2}\right)\sin\alpha = 0\cdot\cos\alpha + 1\cdot\sin\alpha = \sin\alpha = \text{LHS}$$

   b) $\cos\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$

   Proof:
$$\text{RHS} = \sin\left(\frac{\pi}{2}-\alpha\right) = \sin\left(\frac{\pi}{2}\right)\cos\alpha - \cos\left(\frac{\pi}{2}\right)\sin\alpha = 1\cdot\cos\alpha - 0\cdot\sin\alpha = \cos\alpha = \text{LHS}$$

2. Prove each of the following identities.

   a) $\frac{2}{\sin 2x} = \tan x + \cot x$

   Proof:
$$\begin{aligned}\text{LHS} &= \frac{2}{\sin 2x} = \frac{2}{2\sin x\cos x} = \frac{1}{\sin x\cos x} = \frac{\sin^2 x+\cos^2 x}{\sin x\cos x} = \frac{\sin^2 x}{\sin x\cos x} + \frac{\cos^2 x}{\sin x\cos x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \tan x + \cot x = \text{RHS}\end{aligned}$$

   b) $\cos 2x = \frac{\cot x - \tan x}{\cot x + \tan x}$

   Proof:
$$\begin{aligned}\text{RHS} &= \frac{\cot x - \tan x}{\cot x + \tan x} = \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}}{\frac{\cos^2 x + \sin^2 x}{\sin x\cos x}} = \frac{\cos^2 x - \sin^2 x}{\sin x\cos x}\cdot\frac{\sin x\cos x}{\cos^2 x+\sin^2 x}\\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{\cos 2x}{1} = \text{LHS}\end{aligned}$$

   c) $\frac{1+\cos 2x}{1-\cos 2x} = \cot^2 x$

   Proof: Recall that the double-angle formula for cosine has three forms:
$$\cos 2x = \cos^2 x - \sin^2 x \quad\text{and}\quad \cos 2x = 2\cos^2 x - 1 \quad\text{and}\quad \cos 2x = 1 - 2\sin^2 x$$
   We will use different forms for the numerator and denominator.
$$\text{LHS} = \frac{1+\cos 2x}{1-\cos 2x} = \frac{1+2\cos^2 x - 1}{1-\left(1-2\sin^2 x\right)} = \frac{2\cos^2 x}{2\sin^2 x} = \frac{\cos^2 x}{\sin^2 x} = \cot^2 x = \text{RHS}$$

   d) $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

   Proof: This identity can look much less intimidating if we introduce a new variable. Let $\alpha = \frac{x}{2}$. If we multiply both sides by 2, then $2\alpha = x$. Using α instead of x, the identity
$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad\text{becomes}\quad \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

$$\begin{aligned}\text{RHS} &= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{\cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{1}{\cos^2\alpha}}\\ &= \frac{2\sin\alpha}{\cos\alpha}\cdot\frac{\cos^2\alpha}{1} = 2\cos\alpha\sin\alpha = \sin 2\alpha = \sin x = \text{LHS}\end{aligned}$$

e) $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = 1+\sin x$

Proof: This identity can look much less intimidating if we introduce a new variable. Let $\alpha = \frac{x}{2}$. If we multiply both sides by 2, then $2\alpha = x$. Using α instead of x, the identity

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = 1+\sin x \quad \text{becomes} \quad (\sin\alpha+\cos\alpha)^2 = 1+\sin 2\alpha$$

$$\begin{aligned}\text{LHS} &= \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = (\sin\alpha+\cos\alpha)^2 = \sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha = 1+2\sin\alpha\cos\alpha\\ &= 1+\sin 2\alpha = 1+\sin x = \text{RHS}\end{aligned}$$

f) $\dfrac{\tan 2x}{\tan x} = 1+\dfrac{1}{\cos 2x}$

Proof:

$$\begin{aligned}\text{LHS} &= \frac{\tan 2x}{\tan x} = \frac{\frac{2\tan x}{1-\tan^2 x}}{\tan x} = \frac{2\tan x}{1-\tan^2 x}\cdot\frac{1}{\tan x} = \frac{2}{1-\tan^2 x} = \frac{2}{1-\frac{\sin^2 x}{\cos^2 x}} = \frac{2}{\frac{\cos^2 x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}}\\ &= \frac{2}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}} = 2\cdot\frac{\cos^2 x}{\cos^2 x-\sin^2 x} = \frac{2\cos^2 x}{\cos^2 x-\sin^2 x} = \frac{\cos^2 x+\cos^2 x}{\cos^2 x-\sin^2 x}\\ &= \frac{\cos^2 x+\cos^2 x+\sin^2 x-\sin^2 x}{\cos^2 x-\sin^2 x} = \frac{\cos^2 x-\sin^2 x}{\cos^2 x-\sin^2 x}+\frac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x}\\ &= 1+\frac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x} = 1+\frac{1}{\cos 2x} = \text{RHS}\end{aligned}$$

g) $\dfrac{2}{1-\tan^2 x} = \dfrac{1+\cos 2x}{\cos 2x}$

Proof:

$$\text{RHS} = \frac{1+\cos 2x}{\cos 2x} = \frac{1+\cos 2x}{\cos^2 x-\sin^2 x} = \frac{1+2\cos^2 x-1}{\cos^2 x-\sin^2 x} = \frac{2\cos^2 x}{\cos^2 x-\sin^2 x}$$

Next, we will divide both numerator and denominator by $\cos^2 x$.

$$\frac{2\cos^2 x}{\cos^2 x-\sin^2 x} = \frac{\frac{2\cos^2 x}{\cos^2 x}}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}} = \frac{2}{\frac{\cos^2 x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}} = \frac{2}{1-\tan^2 x} = \text{LHS}$$

h) $\dfrac{\tan\left(\dfrac{\pi}{4}-x\right)}{\tan\left(\dfrac{\pi}{4}+x\right)}=\dfrac{1-\sin 2x}{1+\sin 2x}$

Proof:

$$\begin{aligned}
\text{LHS} &= \frac{\tan\left(\frac{\pi}{4}-x\right)}{\tan\left(\frac{\pi}{4}+x\right)}=\frac{\frac{1-\tan x}{1+\tan x}}{\frac{\tan x+1}{1-\tan x}}=\frac{1-\tan x}{1+\tan x}\cdot\frac{1-\tan x}{1+\tan x}=\left(\frac{1-\tan x}{1+\tan x}\right)^2=\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)^2\\
&=\left(\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right)^2=\left(\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}\right)^2=\left(\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}\cdot\frac{\cos x}{\cos x}\right)^2=\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)^2\\
&=\frac{1-\sin 2x}{1+\sin 2x}=\text{RHS}
\end{aligned}$$

i) $1+\tan\alpha\tan\beta=\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta}$

$$\begin{aligned}
\text{LHS} &= 1+\tan\alpha\tan\beta=1+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\\
&=\frac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta}=\text{RHS}
\end{aligned}$$

j) $\sin 35^\circ+\sin 25^\circ=\cos 5^\circ$

$$\begin{aligned}
\sin 35^\circ+\sin 25^\circ &= \sin(30^\circ+5^\circ)+\sin(30^\circ-5^\circ)\\
&= \sin 30^\circ\cos 5^\circ+\cos 30^\circ\sin 5^\circ+\sin 30^\circ\cos 5^\circ-\cos 30^\circ\sin 5^\circ\\
&= 2\sin 30^\circ\cos 5^\circ=2\left(\frac{1}{2}\right)\cos 5^\circ=\cos 5^\circ
\end{aligned}$$

k) $\cos 12^\circ-\cos 48^\circ=\sin 18^\circ$

$$\begin{aligned}
\cos 12^\circ-\cos 48^\circ &= \cos(30^\circ-18^\circ)-\cos(30^\circ+18^\circ)\\
&= \cos 30^\circ\cos 18^\circ+\sin 30^\circ\sin 18^\circ-(\cos 30^\circ\cos 18^\circ-\sin 30^\circ\sin 18^\circ)\\
&= \cos 30^\circ\cos 18^\circ+\sin 30^\circ\sin 18^\circ-\cos 30^\circ\cos 18^\circ+\sin 30^\circ\sin 18^\circ\\
&= 2\sin 30^\circ\sin 18^\circ=2\left(\frac{1}{2}\right)\sin 18^\circ=\sin 18^\circ
\end{aligned}$$

3. Compute the exact value of $\dfrac{1+\tan 15^\circ}{1-\tan 15^\circ}$.

Solution 1: Let us first compute $\tan 15^\circ$.

$$\begin{aligned}
\tan 15^\circ &= \tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\
&=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\cdot\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{3+1-2\sqrt{3}}{2}=\frac{4-2\sqrt{2}}{2}=\frac{2\left(2-\sqrt{3}\right)}{2}=2-\sqrt{3}
\end{aligned}$$

Now our expression becomes

$$\frac{1+\tan 15^\circ}{1-\tan 15^\circ}=\frac{1+\left(2-\sqrt{3}\right)}{1-\left(2-\sqrt{3}\right)}=\frac{3-\sqrt{3}}{-1+\sqrt{3}}=\frac{\sqrt{3}\left(\sqrt{3}-1\right)}{\sqrt{3}-1}=\sqrt{3}$$

Solution 2: (a very neat trick) With a little ingenuity, we can turn this expression into a sum formula for tangent.
We re-write 1 as $\tan 45^\circ$.

$$\frac{1+\tan 15^\circ}{1-\tan 15^\circ}=\frac{1+\tan 15^\circ}{1-1\cdot\tan 15^\circ}=\frac{\tan 45^\circ+\tan 15^\circ}{1-\tan 45^\circ\cdot\tan 15^\circ}=\tan\left(45^\circ+15^\circ\right)=\tan 60^\circ=\sqrt{3}$$

Solution 3: Let x denote 15°

$$\begin{aligned}\frac{1+\tan x}{1-\tan x} &= \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}=\frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}\cdot\frac{\cos x}{\cos x}=\frac{\cos x+\sin x}{\cos x-\sin x}\\ &= \frac{\cos x+\sin x}{\cos x-\sin x}\cdot\frac{\cos x+\sin x}{\cos x+\sin x}=\frac{\cos^2 x+\sin^2 x+2\sin x\cos x}{\cos^2 x-\sin^2 x}=\frac{1+\sin 2x}{\cos 2x}\end{aligned}$$

In this case, $x=15^\circ$ means that

$$\frac{1+\tan 15^\circ}{1-\tan 15^\circ}=\frac{1+\sin 30^\circ}{\cos 30^\circ}=\frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}}=\frac{3}{2}\cdot\frac{2}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}$$

4. Compute the exact value of $\tan\alpha$ if $\sin\left(\alpha+60^\circ\right)=\cos\alpha$

Soluton:

$$\begin{aligned}\sin\left(\alpha+60^\circ\right) &= \cos\alpha\\ \sin\alpha\cos 60^\circ+\cos\alpha\sin 60^\circ &= \cos\alpha\\ \sin\alpha\left(\frac{1}{2}\right)+\cos\alpha\left(\frac{\sqrt{3}}{2}\right) &= \cos\alpha\\ \frac{1}{2}\sin\alpha &= \cos\alpha-\cos\alpha\left(\frac{\sqrt{3}}{2}\right)\\ \frac{1}{2}\sin\alpha &= \left(1-\frac{\sqrt{3}}{2}\right)\cos\alpha\end{aligned}$$

Case 1. If $\cos\alpha=0$, then clearly

$$\begin{aligned}\frac{1}{2}\sin\alpha &= \left(1-\frac{\sqrt{3}}{2}\right)\cdot 0\\ \frac{1}{2}\sin\alpha &= 0\\ \sin\alpha &= 0\end{aligned}$$

That's impossible since $\sin^2\alpha+\cos^2\alpha=1$.

Case 2. If $\cos\alpha\neq 0$, then we can divide by it.

$$\begin{aligned}\frac{1}{2}\sin\alpha &= \left(1-\frac{\sqrt{3}}{2}\right)\cos\alpha && \text{divide by }\cos\alpha\\ \frac{1}{2}\frac{\sin\alpha}{\cos\alpha} &= \left(1-\frac{\sqrt{3}}{2}\right) && \text{mulitply by }2\\ \tan\alpha &= 2\left(1-\frac{\sqrt{3}}{2}\right)=2-\sqrt{3}\end{aligned}$$

5. Prove that if $\sin(x+y) = 3\sin(x-y)$, then $\tan x = 2\tan y$.

Solution:

$$\begin{aligned}
\sin(x+y) &= 3\sin(x-y) \\
\sin x\cos y + \cos x \sin y &= 3\sin x\cos y - 3\cos x\sin y && \text{add } 3\cos x\sin y \\
\sin x\cos y + 4\cos x\sin y &= 3\sin x\cos y && \text{subtract } \sin x\cos y \\
4\cos x\sin y &= 2\sin x\cos y && \text{divide by } \cos x\cos y \\
\frac{4\cos x\sin y}{\cos x\cos y} &= \frac{2\sin x\cos y}{\cos x\cos y} \\
4\tan y &= 2\tan x \\
2\tan y &= \tan x
\end{aligned}$$

6. Given that $\dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{3}{5}$ and $\tan\alpha = \dfrac{1}{2}$, find the exact value of $\tan\beta$.

Solution:

$$\begin{aligned}
\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} &= \frac{3}{5} \\
5\cos(\alpha-\beta) &= 3\cos(\alpha+\beta) \\
5\cos\alpha\cos\beta + 5\sin\alpha\sin\beta &= 3\cos\alpha\cos\beta - 3\sin\alpha\sin\beta && \text{add } 3\sin\alpha\sin\beta \\
5\cos\alpha\cos\beta + 8\sin\alpha\sin\beta &= 3\cos\alpha\cos\beta && \text{subtract } 5\cos\alpha\cos\beta \\
8\sin\alpha\sin\beta &= -2\cos\alpha\cos\beta && \text{divide by } 2\cos\alpha\cos\beta \\
\frac{4\sin\alpha\sin\beta}{\cos\alpha\cos\beta} &= -\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \\
4\tan\alpha\tan\beta &= -1 && \tan\alpha = \frac{1}{2} \\
4\left(\frac{1}{2}\right)\tan\beta &= -1 \\
2\tan\beta &= -1 \\
\tan\beta &= -\frac{1}{2}
\end{aligned}$$

check: $\tan\alpha = \dfrac{1}{2}$ $\quad\sin\alpha = \dfrac{1}{\sqrt{5}}$ $\quad\cos\alpha = \dfrac{2}{\sqrt{5}}$

$\tan\beta = -\dfrac{1}{2}$ $\quad\sin\beta = -\dfrac{1}{\sqrt{5}}$ $\quad\cos\beta = \dfrac{2}{\sqrt{5}}$

$$\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} = \frac{\frac{2}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}\left(-\frac{1}{\sqrt{5}}\right)}{\frac{2}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}\left(-\frac{1}{\sqrt{5}}\right)} = \frac{\frac{4}{5}-\frac{1}{5}}{\frac{4}{5}+\frac{1}{5}} = \frac{\frac{3}{5}}{\frac{5}{5}} = \frac{3}{5}$$

7. Given that $\tan\alpha$ and $\tan\beta$ are the solutions of the equation $x^2+6x+7=0$, show that $\sin(\alpha+\beta) = \cos(\alpha+\beta)$.

Solution: We first solve the quadratic equation.

$$\begin{aligned}
x^2+6x+7 &= 0 \\
x^2+6x+9-9+7 &= 0 \\
(x+3)^2-2 &= 0 \\
(x+3)^2-\left(\sqrt{2}\right)^2 &= 0 \\
\left(x+3+\sqrt{2}\right)\left(x+3-\sqrt{2}\right) &= 0
\end{aligned}$$

$$x_1 = -3 - \sqrt{2} \quad \text{and} \quad x_2 = -3 + \sqrt{2}$$

So let's say that $\tan\alpha = -3 - \sqrt{2}$ and $\tan\beta = -3 + \sqrt{2}$. then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{-3-\sqrt{2} + \left(-3+\sqrt{2}\right)}{1 - \left(-3-\sqrt{2}\right)\left(-3+\sqrt{2}\right)} = \frac{-6}{1-7} = 1$$

and since

$$\begin{aligned} \tan(\alpha+\beta) &= 1 \\ \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} &= 1 \\ \sin(\alpha+\beta) &= \cos(\alpha+\beta) \end{aligned}$$

8. In triangle ABC, $2\cos\beta\sin\gamma = \sin\alpha$. Prove that the triangle is isosceles.

   Solution: $\alpha+\beta+\gamma = 180°$. Let us write α as $180° - (\beta+\gamma)$

$$\begin{aligned} 2\cos\beta\sin\gamma &= \sin\alpha \\ 2\cos\beta\sin\gamma &= \sin(180° - (\beta+\gamma)) \end{aligned}$$

   Recall that for all x, $\sin x = \sin(180° - x)$ and so $\sin(180° - (\beta+\gamma))$ can be simplified as $\sin(\beta+\gamma)$.

$$\begin{aligned} 2\cos\beta\sin\gamma &= \sin(\beta+\gamma) \\ 2\cos\beta\sin\gamma &= \sin\beta\cos\gamma + \cos\beta\sin\gamma \qquad \text{subtract } 2\cos\beta\sin\gamma \\ 0 &= \sin\beta\cos\gamma - \cos\beta\sin\gamma \\ 0 &= \sin(\beta-\gamma) \\ \beta-\gamma &= k\cdot 180° \\ \beta &= \gamma + k\cdot 180° \end{aligned}$$

   Since β and γ are both between $0°$ and $180°$, this means that $k=0$. Thus $\beta=\gamma$ and so $b=c$ and the triangle is isosceles.

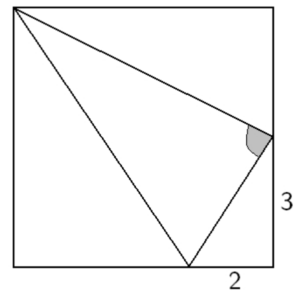
9. Given that $\alpha+\beta+\gamma = 90°$, where α, β, and γ are acute angles. Prove that $\cot\alpha\cot\beta\cot\gamma = \cot\alpha+\cot\beta+\cot\gamma$.

   Solution: $\gamma = 90° - (\beta+\gamma)$ and $\cot\gamma = \cot(90° - (\beta+\gamma))$.

$$\cot(90° - (\alpha+\beta)) = \tan(\alpha+\beta) \text{ by the co-function identities}$$

$$\begin{aligned} \text{RHS} &= \cot\alpha + \cot\beta + \cot\gamma = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \tan(\alpha+\beta) \\ &= \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ &= \frac{\tan\beta(1-\tan\alpha\tan\beta)}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} + \frac{\tan\alpha(1-\tan\alpha\tan\beta)}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} + \frac{\tan\alpha\tan\beta(\tan\alpha+\tan\beta)}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} \\ &= \frac{\tan\beta(1-\tan\alpha\tan\beta) + \tan\alpha(1-\tan\alpha\tan\beta) + \tan\alpha\tan\beta(\tan\alpha+\tan\beta)}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} \\ &= \frac{\tan\beta - \tan\alpha\tan^2\beta + \tan\alpha - \tan^2\alpha\tan\beta + \tan^2\alpha\tan\beta + \tan\alpha\tan^2\beta}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} \\ &= \frac{\tan\beta+\tan\alpha}{\tan\alpha\tan\beta(1-\tan\alpha\tan\beta)} = \frac{1}{\tan\alpha}\cdot\frac{1}{\tan\beta}\cdot\frac{\tan\beta+\tan\alpha}{(1-\tan\alpha\tan\beta)} = \frac{1}{\tan\alpha}\cdot\frac{1}{\tan\beta}\cdot\tan(\alpha+\beta) \\ &= \frac{1}{\tan\alpha}\cdot\frac{1}{\tan\beta}\cdot\cot(90° - (\alpha+\beta)) = \frac{1}{\tan\alpha}\cdot\frac{1}{\tan\beta}\cdot\cot\gamma = \cot\alpha\cot\beta\cot\gamma = \text{LHS} \end{aligned}$$

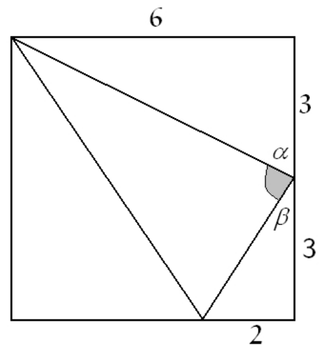
10. Find the exact value of the tangent of the angle shaded on the picture. The quadrilateral is a square with sides 6 units long.



Solution: Let us donte our angle by θ, and the other angles near it by α and β, as shown on the picture. Clearly, $\theta = 180° - (\alpha + \beta)$. Then

$$\tan\theta = \tan(180° - (\alpha+\beta)) = -\tan(\alpha+\beta)$$

where $\tan\alpha = \dfrac{6}{3} = 2$ and $\tan\beta = \dfrac{2}{3}$.



We apply the sum formula for tangent:

$$\tan\theta = -\tan(\alpha+\beta) = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= -\frac{2 + \frac{2}{3}}{1 - 2\cdot\frac{2}{3}} = -\frac{\frac{8}{3}}{-\frac{1}{3}} = 8$$

11. Find the exact value of $\sin 2x$ if we know that $1 + \tan x = \dfrac{35}{12}\sin x$.

Solution:

$$\begin{aligned}
1 + \tan x &= \frac{35}{12}\sin x \\
1 + \frac{\sin x}{\cos x} &= \frac{35}{12}\sin x && \text{multiply by } \cos x \\
\cos x + \sin x &= \frac{35}{12}\sin x\cos x && \text{square both sides} \\
(\cos x + \sin x)^2 &= \left(\frac{35}{12}\sin x\cos x\right)^2 \\
\cos^2 x + \sin^2 x + 2\sin x\cos x &= \left(\frac{35}{12}\right)^2 \sin^2 x\cos^2 x \\
1 + 2\sin x\cos x &= \left(\frac{35}{12}\right)^2 (\sin x\cos x)^2 && \text{let } a = \sin x\cos x \\
1 + 2a &= \frac{35^2}{12^2}a^2 && \text{multiply by 144} \\
144 + 288a &= 1225a^2 \\
0 &= 1225a^2 - 288a - 144
\end{aligned}$$

We solve the quadratic equation using the quadratic formula:

$$a_{1,2} = \frac{288 \pm \sqrt{(-288)^2 - 4(1225)(-144)}}{2 \cdot 1225} = \frac{288 \pm \sqrt{82\,944 + 705\,600}}{2450}$$

$$= \frac{288 \pm \sqrt{788\,544}}{2450} = \frac{288 \pm 888}{2450} = \begin{cases} \dfrac{288 + 888}{2450} = \dfrac{1176}{2450} = \dfrac{12}{25} \\ \dfrac{288 - 888}{2450} = \dfrac{-600}{2450} = -\dfrac{12}{49} \end{cases}$$

Recall that $a = \sin x \cos x$. Then $\sin 2x = 2 \sin x \cos x = 2a = \dfrac{24}{25}$ or $-\dfrac{24}{49}$.

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes.
E-mail questions or comments to mhidegkuti@ccc.edu.

