Sample Problems

1. Express each of the following products as a sum or a difference.

a) $\sin 5x \cos 12x$ c) $\sin 8\alpha \cos 2\alpha$

b) $\cos 24A \cos 8A$ d) $\sin 7\theta \sin 4\theta$

2. Re-write each of the following sums as a product.

a) $\sin 13a - \sin 3a$ b) $\cos 8\beta + \cos 20\beta$

3. Simplify the given expression. $\dfrac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

Practice Problems

1. Express each of the following products as a sum or a difference.

a) $\sin 8A \cos 6A$ c) $\cos 8A \cos 6A$ e) $\sin 3\alpha \cos 19\alpha$ g) $\sin 6m \sin 3m$

b) $\sin 6A \cos 8A$ d) $\sin 8A \sin 6A$ f) $\cos 42\theta \cos 6\theta$ h) $\sin \alpha \cos \beta$

2. Re-write each of the following sums as a product.

a) $\sin 10M + \sin 4M$ b) $\sin 10M - \sin 4M$ c) $\cos 5x + \cos 21x$ d) $\cos 21x - \cos 5x$

3. Simplify the given expression. $\dfrac{\cos 4x - \cos 6x}{\sin 6x - \sin 4x}$

Sample Problems - Answers

1. a) $\frac{1}{2}\sin 17x - \frac{1}{2}\sin 7x$ b) $\frac{1}{2}\cos 32A + \frac{1}{2}\cos 16A$ c) $\frac{1}{2}\sin 10\alpha + \frac{1}{2}\sin 6\alpha$
   d) $\frac{1}{2}\cos 3\theta - \frac{1}{2}\cos 11\theta$
2. a) $2\cos 8a \sin 5a$ b) $2\cos 14\beta \cos 6\beta$ 3. $2\sin 2A$

Practice Problems - Answers

1. a) $\frac{1}{2}\sin 14A + \frac{1}{2}\sin 2A$ b) $\frac{1}{2}\sin 14A - \frac{1}{2}\sin 2A$ c) $\frac{1}{2}\cos 14A + \frac{1}{2}\cos 2A$
   d) $\frac{1}{2}\cos 2A - \frac{1}{2}\cos 14A$ e) $\frac{1}{2}\sin 22\alpha - \frac{1}{2}\sin 16\alpha$ f) $\frac{1}{2}\cos 36\theta + \frac{1}{2}\cos 48\theta$
   g) $\frac{1}{2}\cos 3m - \frac{1}{2}\cos 9m$ h) $\frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$
2. a) $2\sin 7M \cos 3M$ b) $2\sin 3M \cos 7M$ c) $2\cos 13x \cos 8x$ d) $-2\sin 13x \sin 8x$
3. $\tan 5x$

Sample Problems - Solutions

1. Express each of the following products as a sum or a difference.

a) $\sin 5x \cos 12x$

We will write the sum and difference formula for sine using $5x$ and $12x$.

$$\begin{aligned} \sin 17x &= \sin(12x + 5x) = \sin 12x \cos 5x + \cos 12x \sin 5x \\ \sin 7x &= \sin(12x - 5x) = \sin 12x \cos 5x - \cos 12x \sin 5x \end{aligned}$$

We will subtract the second equation from the first one - or rather, add the opposite.

$$\begin{array}{rl} & \sin 17x = \sin 12x \cos 5x + \cos 12x \sin 5x \\ + & -\sin 7x = -\sin 12x \cos 5x + \cos 12x \sin 5x \\ \hline \end{array}$$

$$\sin 17x - \sin 7x = 2 \cos 12x \sin 5x$$

$$\frac{1}{2}(\sin 17x - \sin 7x) = \cos 12x \sin 5x$$

and so the answer is $\cos 12x \sin 5x = \boxed{\frac{1}{2}\sin 17x - \frac{1}{2}\sin 7x}$

b) $\cos 24A \cos 8A$

We will write the sum and difference formula for cosine using $8A$ and $24A$.

$$\begin{aligned} \cos 32A &= \cos(24A + 8A) = \cos 24A \cos 8A - \sin 24A \sin 8A \\ \cos 16A &= \cos(24A - 8A) = \cos 24A \cos 8A + \sin 24A \sin 8A \end{aligned}$$

We will add the two equations

$$\begin{array}{rl} & \cos 32A = \cos 24A \cos 8A - \sin 24A \sin 8A \\ + & \cos 16A = \cos 24A \cos 8A + \sin 24A \sin 8A \\ \hline \end{array}$$

$$\cos 32A + \cos 16A = 2 \cos 24A \cos 8A$$

$$\frac{1}{2}(\cos 32A + \cos 16A) = \cos 24A \cos 8A$$

and so the answer is $\cos 24A \cos 8A = \boxed{\frac{1}{2}\cos 32A + \frac{1}{2}\cos 16A}$

c) $\sin 8\alpha \cos 2\alpha$

We will write the sum and difference formula for sine using $8M$ and $2M$.

$$\begin{aligned} \sin 10\alpha &= \sin(8\alpha + 2\alpha) = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha \\ \sin 6\alpha &= \sin(8\alpha - 2\alpha) = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha \end{aligned}$$

We will add the two equations.

$$\begin{array}{rl} & \sin 10\alpha = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha \\ + & \sin 6\alpha = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha \\ \hline \end{array}$$

$$\sin 10\alpha + \sin 6\alpha = 2 \sin 8\alpha \cos 2\alpha$$

$$\frac{1}{2}(\sin 10\alpha + \sin 6\alpha) = \sin 8\alpha \cos 2\alpha$$

and so the answer is $\sin 8\alpha \cos 2\alpha = \boxed{\frac{1}{2}(\sin 10\alpha + \sin 6\alpha)}$

d) $\sin 7\theta \sin 4\theta$

We will write the sum and difference formula for cosine using $8A$ and $24A$.

$$\begin{aligned}\cos 11\theta &= \cos(7\theta + 4\theta) = \cos 7\theta \cos 4\theta - \sin 7\theta \sin 4\theta \\ \cos 3\theta &= \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta\end{aligned}$$

We will subtract the first equation from the second - or rather, add the opposite:

$$\begin{array}{rl} -\cos 11\theta = -(\cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta) & \\ + \quad \cos 3\theta = \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta & \\ \hline \cos 3\theta - \cos 11\theta = 2\sin 7\theta \sin 4\theta & \\ \sin 7\theta \sin 4\theta = \frac{1}{2}(\cos 3\theta - \cos 110) & \end{array}$$

and so the answer is $\sin 7\theta \sin 4\theta = \boxed{\frac{1}{2}(\cos 3\theta - \cos 11\theta)}$

2. Re-write each of the following sums as a product.

a) $\sin 13a - \sin 3a$

Solution: We will need to come up with two angles that add up to $13a$ and their difference is $3a$. This is quite easy, especially if we just remember that one of those angles must be the average of $13a$ and $3a$. This average is $\frac{13a + 3a}{2} = 8a$. Now the other angle is $13a - 8a = 5a$. Indeed, the sum of $5a$ and $8a$ is $13a$ and their difference is $3a$. There are no more decisions to make. Looking at the problem posed, we know that we will express the sum and difference formulas for sine; and that we will subtract $\sin 3a$ from $\sin 13a$.

$$\begin{aligned}\sin 13a &= \sin(8a + 5a) = \sin 8a \cos 5a + \cos 8a \sin 5a \\ \sin 3a &= \sin(8a - 5a) = \sin 8a \cos 5a - \cos 8a \sin 5a\end{aligned}$$

We will subtract $\sin 3a$ from $\sin 13a$ - or rather, add the opposite:

$$\begin{array}{rl} \sin 13a = \sin(8a + 5a) = \quad \sin 8a \cos 5a \quad + \quad \cos 8a \sin 5a & \\ + \quad -\sin 3a = -\sin(8a - 5a) = -\sin 8a \cos 5a + \cos 8a \sin 5a & \\ \hline \sin 13a - \sin 3a = 2\cos 8a \sin 5a & \end{array}$$

and so the answer is $\sin 13a - \sin 3a = \boxed{2\cos 8a \sin 5a}$

b) $\cos 8\beta + \cos 20\beta$

Solution: We will need to come up with two angles that add up to 20β and their difference is 8β. This is quite easy, especially if we just remember that one of those angles must be the average these two, $\dfrac{8\beta + 20\beta}{2} = 14\beta$. Now the other angle is $20\beta - 14\beta = 6\beta$. Looking at the problem given, we know that we will express the sum and difference formulas for cosine; and that we will add $\cos 8\beta$ and $\cos 20\beta$.

$$\begin{aligned}
\cos 20\beta &= \sin(14\beta + 6\beta) = \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\
\cos 8\beta &= \sin(14\beta - 6\beta) = \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta
\end{aligned}$$

We will add the two equations

$$\begin{array}{rl}
 & \cos 20\beta = \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\
+ & \cos 8\beta = \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta \\
\hline
 & \cos 8\beta + \cos 20\beta = 2\cos 14\beta \cos 6\beta
\end{array}$$

and so the answer is $\cos 8\beta + \cos 20\beta = \boxed{2\cos 14\beta \cos 6\beta}$

3. Simplify the given expression. $\dfrac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

Solution: Let us use the sum-product identities to turn the numerator and denominator into products. Then, if there is a common factor, we can cancel it out.

If we need to find two quantities with sum $11A$ and difference $3A$, let's take the average of the two first. We obtain $7A$ and then easily $4A$. Now we will state the sum and difference formula of cosine with angles $7A$ and $11A$.

$$\cos 11A = \cos(7A + 4A) = \cos 7A \cos 4A - \sin 7A \sin 4A$$

$$\cos 3A = \cos(7A - 4A) = \cos 7A \cos 4A + \sin 7A \sin 4A$$

Since we want to express $\cos 3A - \cos 11A$, we subtract the equations accordingly.

$$\begin{array}{rl}
 & -\cos 11A = -\cos 7A \cos 4A + \sin 7A \sin 4A \\
+ & \cos 3A = \cos 7A \cos 4A + \sin 7A \sin 4A \\
\hline
 & \cos 3A - \cos 11A = 2\sin 7A \sin 4A
\end{array}$$

so the numerator can be re-written as $2\sin 7A \sin 4A$.

We repeat the process with the denominator. First we need to find two quantities with sum $9A$ and difference $5A$. We find $7A$ and then easily $2A$. Now we will state the sum and difference formula of sine with angles $7A$ and $2A$.

$$\sin 9A = \sin(7A + 2A) = \sin 7A \cos 2A + \cos 7A \sin 2A$$

$$\sin 5A = \sin(7A - 2A) = \sin 7A \cos 2A - \cos 7A \sin 2A$$

We add the two equations and obtain $\sin 9A + \sin 5A = 2\sin 7A \cos 2A$. So the denominator can be re-written as $2\sin 7A \cos 2A$. So our fraction is now

$$\frac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A} = \frac{2\sin 7A \sin 4A}{2\sin 7A \cos 2A} = \frac{\sin 4A}{\cos 2A}$$

This expression can be further simplified, via the double-angle formula for sine: $\sin 4A = 2\sin 2A \cos 2A$.

$$\frac{\sin 4A}{\cos 2A} = \frac{2\sin 2A \cos 2A}{\cos 2A} = 2\sin 2A$$

So the answer is $\boxed{2\sin 2A}$.

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