1. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ b) $\cos^{-1}\left(-\frac{1}{2}\right)$ c) $\tan^{-1}\left(-\sqrt{3}\right)$ d) $\sin^{-1}(0)$ e) $\cos^{-1}(0)$ f) $\sin^{-1}(-1)$ g) $\tan^{-1}(1)$

2. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$
b) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$
c) $\cos^{-1}\left(\cos\frac{5\pi}{6}\right)$
d) $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$
e) $\cos^{-1}(\cos 3\pi)$
f) $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
g) $\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$
h) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$
i) $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$
j) $\sin^{-1}(\sin(3\pi))$
k) $\tan^{-1}(\tan(0))$
l) $\tan^{-1}(\tan(\pi))$
m) $\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)$
n) $\tan^{-1}\left(\tan\left(\frac{11\pi}{4}\right)\right)$

3. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
b) $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$
c) $\cos^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
d) $\cos^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$
e) $\cos^{-1}(\sin(3\pi))$
f) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$
g) $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$
h) $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$
i) $\tan\left(\cos^{-1}(-1)\right)$

4. Find the exact value of each of the follwing.

a) $\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$
b) $\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$
c) $\tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$
d) $\cos\left(\tan^{-1}(2)\right)$
e) $\cos\left(\tan^{-1}(-2)\right)$
f) $\sin\left(\tan^{-1}(-2)\right)$

5. Simplify each of the follwing.

a) $\cos\left(\sin^{-1}x\right)$ b) $\tan\left(\sin^{-1}x\right)$ c) $\tan\left(\cos^{-1}x\right)$ d) $\cos\left(\tan^{-1}x\right)$

6. Compute the exact value for each of the following.

a) $\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$
b) $\cos\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)$
c) $\tan\left(2\tan^{-1}\left(\frac{2}{5}\right)\right)$
d) $\sin\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$
e) $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(-\frac{5}{13}\right)\right)$
f) $\cos\left(\sin^{-1}\left(-\frac{1}{3}\right)-\cos^{-1}\left(\frac{2}{3}\right)\right)$
g) $\tan\left(\tan^{-1}(3)+\tan^{-1}(7)\right)$
h) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{3}{4}\right)\right)$

Answers

1. a) $\dfrac{\pi}{4}$ b) $\dfrac{2\pi}{3}$ c) $-\dfrac{\pi}{3}$ d) 0 e) $\dfrac{\pi}{2}$ f) $-\dfrac{\pi}{2}$ g) $\dfrac{\pi}{4}$

2. a) $\dfrac{\pi}{6}$ b) $\dfrac{\pi}{6}$ c) $\dfrac{5\pi}{6}$ d) $\dfrac{\pi}{2}$ e) π f) $\dfrac{\pi}{6}$ g) $-\dfrac{\pi}{6}$ h) $\dfrac{\pi}{6}$ i) $-\dfrac{\pi}{2}$ j) 0 k) 0 l) 0 m) $\dfrac{\pi}{3}$ n) $-\dfrac{\pi}{4}$

3. a) $\dfrac{\pi}{3}$ b) $\dfrac{2\pi}{3}$ c) $\dfrac{\pi}{3}$ d) π e) $\dfrac{\pi}{2}$ f) $\dfrac{\sqrt{3}}{2}$ g) $\dfrac{1}{2}$ h) $-\dfrac{\sqrt{3}}{3}$ i) 0

4. a) $\dfrac{1}{3}$ b) $\dfrac{2\sqrt{2}}{3}$ c) $-\dfrac{2\sqrt{21}}{21}$ d) $\dfrac{\sqrt{5}}{5}$ e) $\dfrac{\sqrt{5}}{5}$ f) $-\dfrac{2\sqrt{5}}{5}$

5. a) $\sqrt{1-x^2}$ b) $\dfrac{x}{\sqrt{1-x^2}}$ c) $\dfrac{1}{x}\sqrt{1-x^2}$ d) $\dfrac{1}{\sqrt{x^2+1}}$

6. a) $\dfrac{24}{25}$ b) $-\dfrac{7}{9}$ c) $\dfrac{20}{21}$ d) $\dfrac{6\sqrt{7}}{16}$ e) $\dfrac{33}{65}$ f) $\dfrac{4\sqrt{2}-\sqrt{5}}{9}$ g) $-\dfrac{1}{2}$ h) $\dfrac{\sqrt{7}-3\sqrt{3}}{8}$

Sample Problems - Solutions

1. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Solution: We first solve the equation $\cos x = \frac{1}{\sqrt{2}}$. The solution is $x = \pm\frac{\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{\pi}{4}$. Thus $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

b) $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution: We first solve the equation $\cos x = -\frac{1}{2}$. The solution is $x = \pm\frac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{2\pi}{3}$. Thus $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

c) $\tan^{-1}\left(-\sqrt{3}\right)$

Solution: We first solve the equation $\tan x = -\sqrt{3}$. The solution is $x = -\frac{\pi}{3} + k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}\left(-\sqrt{3}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$. This value is $-\frac{\pi}{3}$. Thus $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

d) $\sin^{-1}(0)$

Solution: We first solve the equation $\sin x = 0$. The solution is $x = k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1} 0$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is 0. Thus $\sin^{-1} 0 = 0$.

e) $\cos^{-1}(0)$

Solution: We first solve the equation $\cos x = 0$. The solution is $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1} 0$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{\pi}{2}$. Thus $\cos^{-1} 0 = \frac{\pi}{2}$.

f) $\sin^{-1}(-1)$

Solution: We first solve the equation $\sin x = -1$. The solution is $x = \frac{3\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}(-1)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $-\frac{\pi}{2}$. Thus $\sin^{-1}(-1) = -\frac{\pi}{2}$.

g) $\tan^{-1}(1)$

Solution: We first solve the equation $\tan x = 1$. The solution is $x = \frac{\pi}{4} + k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}(1)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$. This value is $\frac{\pi}{4}$. Thus $\tan^{-1}(1) = \frac{\pi}{4}$.

2. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$

Solution: Our first guess might be $\frac{\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \cos x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will work. First, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. So, $\cos^{-1}\left(\cos\frac{\pi}{6}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

As in the previous problem, we solve $\cos x = \frac{\sqrt{3}}{2}$. The solution is $x = \pm\frac{\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$.

This value is $\frac{\pi}{6}$. Thus $\cos^{-1}\left(\cos\frac{\pi}{6}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

b) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$

Solution: Our first guess might be $-\frac{\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \cos x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. As we will see, in this case, it will NOT work. First, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. So, $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

We solve $\cos x = \frac{\sqrt{3}}{2}$. The solution is $x = \pm\frac{\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$.
This value is $\frac{\pi}{6}$. Thus $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

c) $\cos^{-1}\left(\cos\frac{5\pi}{6}\right)$

Solution: Our first guess might be $\frac{5\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \cos x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. As we will see, in this case, it will work. First, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. So, $\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

We solve $\cos x = -\frac{\sqrt{3}}{2}$. The solution is $x = \pm\frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$.
This value is $\frac{5\pi}{6}$. Thus $\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d) $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$

Solution: Our first guess might be $\frac{3\pi}{2}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \cos x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. As we will see, in this case, it will NOT work. First, $\cos\left(\frac{3\pi}{2}\right) = 0$. So, $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \cos^{-1} 0$.

We solve the equation $\cos x = 0$. The solution is $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1} 0$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{\pi}{2}$. Thus $\cos^{-1} 0 = \frac{\pi}{2}$.. Thus $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \cos^{-1}(0) = \frac{\pi}{2}$.

e) $\cos^{-1}(\cos 3\pi)$

Solution: Our first guess might be 3π because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \cos x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. As we will see, in this case, it will NOT work. First, $\cos(3\pi) = -1$. So, $\cos^{-1}(\cos(3\pi)) = \cos^{-1}(-1)$.

We solve the equation $\cos x = -1$. The solution is $x = \pi + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}(-1)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is π. Thus $\cos^{-1}(-1) = \pi$.. Thus $\cos^{-1}(\cos(3\pi)) = \cos^{-1}(-1) = \pi$.

f) $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

Solution: Our first guess might be $\frac{\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \sin x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will work. First, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. So, $\sin^{-1}\left(\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right)$.

We solve the equation $\sin x = \frac{1}{2}$. The solution is $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}\left(\frac{1}{2}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $\frac{\pi}{6}$. Thus $\sin^{-1}\left(\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

g) $\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Solution: Our first guess might be $-\frac{\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \sin x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will work. First, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$. So, $\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right)$.

We solve the equation $\sin x = -\frac{1}{2}$. The solution is $x = -\frac{\pi}{6} + 2k\pi$ or $x = -\frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $-\frac{\pi}{6}$. Thus $\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

h) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Solution: Our first guess might be $\frac{5\pi}{6}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \sin x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will NOT work. First, $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$. So, $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right)$.

We solve the equation $\sin x = \frac{1}{2}$. The solution is $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}\left(\frac{1}{2}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $\frac{\pi}{6}$. Thus $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

i) $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$

Solution: Our first guess might be $\frac{3\pi}{2}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \sin x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will NOT work. First, $\sin\left(\frac{3\pi}{2}\right) = -1$. So, $\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}(-1)$.

We solve the equation $\sin x = -1$. The solution is $x = -\frac{\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}(-1)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$.
This value is $-\frac{\pi}{2}$. Thus $\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$.

j) $\sin^{-1}(\sin(3\pi))$

Solution: Our first guess might be 3π because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \sin x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will NOT work. First, $\sin(3\pi) = 0$. So, $\sin^{-1}(\sin 3\pi) = \sin^{-1}(0)$.

We solve the equation $\sin x = 0$. The solution is $x = k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1} 0$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is 0. Thus $\sin^{-1}(\sin 3\pi) = \sin^{-1}(0) = 0$.

k) $\tan^{-1}(\tan 0)$

Solution: Our first guess might be 0 because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \tan x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will work. First, $\tan 0 = 0$. So, $\tan^{-1}(\tan 0) = \tan^{-1}(0)$.

We solve the equation $\tan x = 0$. The solution is $x = k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}(0)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$.
This value is 0. Thus $\tan^{-1}(\tan 0) = \tan^{-1}(0) = 0$.

l) $\tan^{-1}(\tan \pi)$

Solution: Our first guess might be π because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \tan x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will NOT work. First, $\tan \pi = 0$. So, $\tan^{-1}(\tan \pi) = \tan^{-1}(0)$.

We solve the equation $\tan x = 0$. The solution is $x = k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}(0)$ is the unique value that falls between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ - $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$.
This value is 0. Thus $\tan^{-1}(\tan \pi) = \tan^{-1}(0) = 0$.

m) $\tan^{-1}\left(\tan\left(\dfrac{\pi}{3}\right)\right)$

Solution: Our first guess might be $\dfrac{\pi}{3}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \tan x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will work. First, $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$. So, $\tan^{-1}\left(\tan\left(\dfrac{\pi}{3}\right)\right) = \tan^{-1}\left(\sqrt{3}\right)$.

We solve the equation $\tan x = \sqrt{3}$. The solution is $x = \dfrac{\pi}{3} + k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}\left(\sqrt{3}\right)$ is the unique value that falls between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ - $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$.
This value is $\dfrac{\pi}{3}$. Thus $\tan^{-1}\left(\tan\left(\dfrac{\pi}{3}\right)\right) = \tan^{-1}\left(\sqrt{3}\right) = \dfrac{\pi}{3}$.

n) $\tan^{-1}\left(\tan\left(\dfrac{11\pi}{4}\right)\right)$

Solution: Our first guess might be $\dfrac{11\pi}{4}$ because we are composing a function and its inverse. However, the identity $f^{-1}(f(x)) = x$ is necessarily true only if f is one-to-one. The function $f(x) = \tan x$ is not one-to-one, so $f^{-1}(f(x)) = x$ may not be true. In this case, it will NOT work. First, $\tan\left(\dfrac{11\pi}{4}\right) = -1$. So, $\tan^{-1}\left(\tan\left(\dfrac{11\pi}{4}\right)\right) = \tan^{-1}(-1)$.

We solve the equation $\tan x = -1$. The solution is $x = -\dfrac{\pi}{4} + k\pi$ where $k \in \mathbb{Z}$. The value of $\tan^{-1}(-1)$ is the unique value that falls between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ - $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is the range of $f(x) = \tan^{-1} x$.
This value is $-\dfrac{\pi}{4}$. Thus $\tan^{-1}\left(\tan\left(\dfrac{11\pi}{4}\right)\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$.

3. Simplify each of the following. Use exact values, and present angles in radians.

a) $\cos^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right)$

Solution: First, $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$. So, $\cos^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right) = \cos^{-1}\left(\dfrac{1}{2}\right)$.

We solve the equation $\cos x = \dfrac{1}{2}$. The solution is $x = \pm\dfrac{\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(\dfrac{1}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\dfrac{\pi}{3}$.
Thus $\cos^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right) = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$.

b) $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Solution: First, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$. So, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right)$.

We solve the equation $\cos x = -\frac{1}{2}$. The solution is $x = \pm\frac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{2\pi}{3}$.
Thus $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

c) $\cos^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

Solution: First, $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$. So, $\cos^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right)$.

We solve the equation $\cos x = \frac{1}{2}$. The solution is $x = \pm\frac{\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(\frac{1}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{\pi}{3}$.
Thus $\cos^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

d) $\cos^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$

Solution: First, $\sin\left(\frac{3\pi}{2}\right) = -1$. So, $\cos^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = \cos^{-1}(-1)$.

We solve the equation $\cos x = -1$. The solution is $x = \pi + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}(-1)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is π.
Thus $\cos^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = \cos^{-1}(-1) = \pi$.

e) $\cos^{-1}(\sin(3\pi))$

Solution: First, $\sin(3\pi) = 0$. So, $\cos^{-1}(\sin(3\pi)) = \cos^{-1}(0)$.

We solve the equation $\cos x = 0$. The solution is $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}(0)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{\pi}{2}$.
Thus $\cos^{-1}(\sin(3\pi)) = \cos^{-1}(0) = \frac{\pi}{2}$.

f) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

Solution: First we find $\cos^{-1}\left(-\frac{1}{2}\right)$. We solve the equation $\cos x = -\frac{1}{2}$. The solution is $x = \pm\frac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is $\frac{2\pi}{3}$. So, $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

g) $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Solution: First we find $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. We solve the equation $\sin x = -\frac{\sqrt{3}}{2}$. The solution is $x = -\frac{\pi}{3} + 2k\pi$ or $x = -\frac{2\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $-\frac{\pi}{3}$. So, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ and so $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$.

h) $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution: First we find $\sin^{-1}\left(-\frac{1}{2}\right)$. We solve the equation $\sin x = -\frac{1}{2}$. The solution is $x = -\frac{\pi}{6} + 2k\pi$ or $x = -\frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is the unique value that falls between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ - $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the range of $f(x) = \sin^{-1} x$. This value is $-\frac{\pi}{6}$. So, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ and so $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

i) $\tan\left(\cos^{-1}(-1)\right)$

Solution: First we find $\cos^{-1}(-1)$. We solve the equation $\cos x = -1$. The solution is $x = \pi + 2k\pi$ where $k \in \mathbb{Z}$. The value of $\cos^{-1}(-1)$ is the unique value that falls between 0 and π - $[0, \pi]$ is the range of $f(x) = \cos^{-1} x$. This value is π. So, $\tan\left(\cos^{-1}(-1)\right) = \tan(\pi) = 0$.

4. Find the exact value of each of the follwing.

a) $\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$

Solution: These problems are different because $\cos^{-1}\left(\frac{1}{3}\right)$ is unlike $\cos^{-1}\left(\frac{1}{2}\right)$. While $\cos^{-1}\left(\frac{1}{2}\right)$ can be simplified as $\frac{\pi}{3}$, we can't do the same with $\cos^{-1}\left(\frac{1}{3}\right)$ because it is not one of the algebraically approachable angles. yet, we need to find the exact value. So, we need to use other techniques here from what we used before.

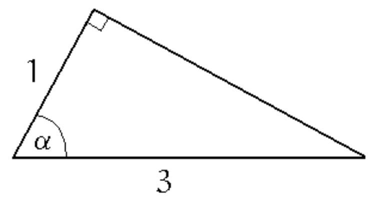
Recall the theorem that $f\left(f^{-1}(x)\right) = x$ for all x. This implies that $\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \frac{1}{3}$.

b) $\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$

Solution: We will first introduce a new variable. Let $\alpha=\cos^{-1}\left(\frac{1}{3}\right)$. This means that α is in $[0,\pi]$ and $\cos\alpha=\frac{1}{3}$. We are asked to find the exact value of $\sin\alpha$.

First we will find the absolute value of $\sin\alpha$. Then we will figure out the sign of $\sin\alpha$.

For the absolute value of $\sin\alpha$, recall that $\cos\alpha=\frac{1}{3}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{8}$. Thus $|\sin\alpha|=\frac{\sqrt{8}}{3}$, or $\sin\alpha=\pm\frac{\sqrt{8}}{3}$



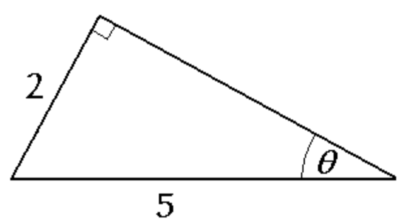
We now consider the sign of $\sin\alpha$. Because α is in the range of $f(x)=\cos^{-1}x$, α is in $[0,\pi]$. Because $\cos\alpha$ is positive, α is in the first quadrant, and so $\sin\alpha$ is positive. Thus the answer is $\frac{\sqrt{8}}{3}=\frac{2\sqrt{2}}{3}$.

c) $\tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$

Solution: We will first introduce a new variable. Let $\theta=\sin^{-1}\left(-\frac{2}{5}\right)$. This means that θ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\theta=-\frac{2}{5}$. We are asked to find the exact value of $\tan\theta$.

First we will find the absolute value of $\tan\theta$. Then we will figure out the sign of $\tan\theta$.

For the absolute value of $\tan\theta$, let us ignore the negative sign for now and assume that $\sin\theta=\frac{2}{5}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{21}$. Thus $|\tan\theta|=\frac{2}{\sqrt{21}}=\frac{2\sqrt{21}}{21}$, or $\tan\theta=\pm\frac{2\sqrt{21}}{21}$



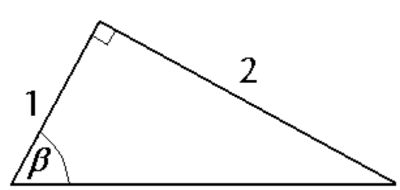
Consider now the sign of $\tan\theta$. Because θ is in the range of $f(x)=\sin^{-1}x$, θ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because $\sin\theta$ is negative, θ is in the fourth quadrant, and so $\tan\theta$ is also negative. Thus the answer is $-\frac{2\sqrt{21}}{21}$.

d) $\cos\left(\tan^{-1}(2)\right)$

Solution: We will first introduce a new variable. Let $\beta=\tan^{-1}(2)$. This means that β is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\tan\beta=2$. We are asked to find the exact value of $\cos\beta$.

First we will find the absolute value of $\cos\beta$. Then we will figure out the sign of $\cos\beta$.

For the absolute value of $\cos\beta$, recall that $\tan\beta=2$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{5}$. Thus $|\cos\beta|=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$, or $\cos\beta=\pm\frac{\sqrt{5}}{5}$



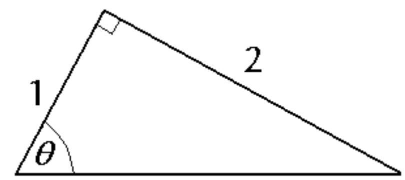
We now consider the sign of $\cos\beta$. Because β is in the range of $f(x) = \tan^{-1} x$, β is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Because $\tan\beta$ is positive, β is in the first quadrant, and so $\cos\beta$ is also positive. Thus the answer is $\frac{\sqrt{5}}{5}$.

e) $\cos\left(\tan^{-1}(-2)\right)$

Solution: We will first introduce a new variable. Let $\theta = \tan^{-1}(-2)$. This means that θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = -2$. We are asked to find the exact value of $\cos\theta$.

First we will find the absolute value of $\cos\theta$. Then we will figure out the sign of $\cos\theta$.

For the absolute value of $\cos\theta$, let us ignore the negative sign for now and assume that $\tan\theta = 2$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{5}$. Thus $|\cos\theta| = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$, or $\cos\theta = \pm\frac{\sqrt{5}}{5}$.



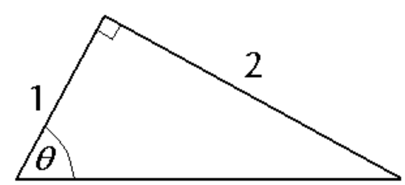
We now consider the sign of $\cos\theta$. Because θ is in the range of $f(x) = \tan^{-1} x$, θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Because $\tan\theta$ is negative, θ is in the fourth quadrant, and so $\cos\theta$ is positive. Thus the answer is $\frac{\sqrt{5}}{5}$

f) $\sin\left(\tan^{-1}(-2)\right)$

Solution: We will first introduce a new variable. Let $\theta = \tan^{-1}(-2)$. This means that θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = -2$. We are asked to find the exact value of $\sin\theta$.

First we will find the absolute value of $\sin\theta$. Then we will figure out the sign of $\sin\theta$.

For the absolute value of $\sin\theta$, let us ignore the negative sign for now and assume that $\tan\theta = 2$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{5}$. Thus $|\sin\theta| = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$, or $\sin\theta = \pm\frac{2\sqrt{5}}{5}$.



We now consider the sign of $\sin\theta$. Because θ is in the range of $f(x) = \tan^{-1} x$, θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Because $\tan\theta$ is negative, θ is in the fourth quadrant, and so $\tan\theta$ is also negative. Thus the answer is $-\frac{2\sqrt{5}}{5}$.

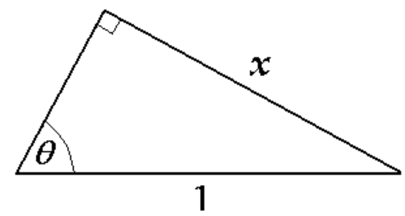
5. Simplify each of the follwing.

a) $\cos\left(\sin^{-1} x\right)$

Solution 1 (Geometric approach) We will first introduce a new variable. Let $\theta = \sin^{-1} x$. This means that θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$. (This also means that $-1 \le x \le 1$). We are asked to find the exact value of $\cos\theta$.

We will first find the absolute value of $\cos\theta$. Then we will figure out the sign of $\cos\theta$.

For the absolute value of $\sin\theta$, let us assume that x is positive and think of it as $\frac{x}{1}$. $\sin\theta = x = \frac{x}{1}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{1-x^2}$. Thus $|\cos\theta| = \sqrt{1-x^2}$, or $\cos\theta = \pm\sqrt{1-x^2}$.



Consider now the sign of $\cos\theta$. Because θ is in the range of $f(x) = \sin^{-1} x$, θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If $x = \sin\theta$ is negative, then θ is in the fourth quadrant. If $x = \sin\theta$ is positive, then θ is in the first quadrant. In both cases, the cosine is positive and so $\cos\theta = \sqrt{1-x^2}$.

Solution 2 (Algebraic approach) We will first introduce a new variable. Let $\theta = \sin^{-1} x$. This means that θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$. (This also means that $-1 \leq x \leq 1$). We are asked to find the exact value of $\cos\theta$.

Recall the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$. Then

$$\begin{aligned} \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1-\sin^2\theta} \end{aligned}$$

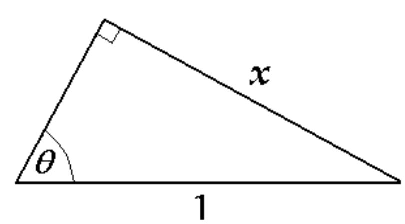
We now consider the sign of $\cos\theta$. Because θ is in the range of $f(x) = \sin^{-1} x$, θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If $x = \sin\theta$ is negative, then θ is in the fourth quadrant. If $x = \sin\theta$ is positive, then θ is in the first quadrant. In both cases, the cosine is positive and so $\cos\theta = \sqrt{1-x^2}$.

b) $\tan\left(\sin^{-1} x\right)$

Solution 1 (Geometric approach) We will first introduce a new variable. Let $\theta = \sin^{-1} x$. This means that θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$. (This also means that $-1 \leq x \leq 1$). We are asked to find the exact value of $\tan\theta$.

We will first find the absolute value of $\tan\theta$. Then we will figure out the sign of $\tan\theta$.

For the absolute value of $\tan\theta$, let us assume that x is positive and think of it as $\frac{x}{1}$. $\sin\theta = x = \frac{x}{1}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{1-x^2}$. Thus $|\tan\theta| = \frac{x}{\sqrt{1-x^2}}$, or $\tan\theta = \pm\frac{x}{\sqrt{1-x^2}}$.



Let us consider the sign of $\tan\theta$ next. Because θ is in the range of $f(x) = \sin^{-1} x$, θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If $x = \sin\theta$ is negative, then θ is in the fourth quadrant. Then $\tan\theta$ is also negative. If $x = \sin\theta$ is positive, then θ is in the first quadrant, and then $\tan\theta$ is also positive. Considering the expression $\frac{x}{\sqrt{1-x^2}}$, we find that the denominator is positive for all values of x, and the numerator is positive if x is positive and negative if x is negative. This is exactly the sign we need and so $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

Solution 2 (Algebraic approach) We will first introduce a new variable. Let $\theta = \sin^{-1} x$. This means that θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$. (This also means that $-1 \leq x \leq 1$). We are asked to find the exact value of $\tan\theta$.

First we will find the absolute value of $\tan\theta$. Then we will figure out the sign of $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}} = \pm\frac{x}{\sqrt{1-x^2}}$$

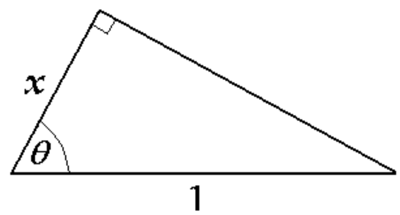
Let us consider the sign of $\tan\theta$ next. Because θ is in the range of $f(x) = \sin^{-1} x$, θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If $x = \sin\theta$ is negative, then θ is in the fourth quadrant. Then $\tan\theta$ is also negative. If $x = \sin\theta$ is positive, then θ is in the first quadrant, and then $\tan\theta$ is also positive. Considering the expression $\frac{x}{\sqrt{1-x^2}}$, we find that the denominator is positive for all values of x, and the numerator is positive if x is positive and negative if x is negative. This is exactly the sign we need and so $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

c) $\tan\left(\cos^{-1} x\right)$

Solution 1. (Geometric approach) We will first introduce a new variable. Let $\theta = \cos^{-1} x$. This means that θ is in $[0, \pi]$ and $\cos\theta = x$. (This also means that $-1 \le x \le 1$). We are asked to find the exact value of $\tan\theta$.

First we will find the absolute value of $\tan\theta$. Then we will figure out the sign of $\tan\theta$.

For the absolute value of $\tan\theta$, let us assume that x is positive and think of it as $\frac{x}{1}$. $\cos\theta = x = \frac{x}{1}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{1-x^2}$. Thus $|\tan\theta| = \frac{\sqrt{1-x^2}}{x}$, or $\tan\theta = \pm\frac{\sqrt{1-x^2}}{x}$.



Let us consider the sign of $\tan\theta$ next. Because θ is in the range of $f(x) = \cos^{-1} x$, θ is in $[0, \pi]$. If $x = \cos\theta$ is negative, then θ is in the second quadrant. Then $\tan\theta$ is also negative. If $x = \cos\theta$ is positive, then θ is in the first quadrant, and then $\tan\theta$ is also positive. Considering the expression $\frac{\sqrt{1-x^2}}{x}$, we find that the numerator is positive for all values of x, and the denominator is positive if x is positive and negative if x is negative. This is exactly the sign we need and so $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

Solution 2. (Algebraic approach) We will first introduce a new variable. Let $\theta = \cos^{-1} x$. This means that θ is in $[0, \pi]$ and $\cos\theta = x$. (This also means that $-1 \le x \le 1$). We are asked to find the exact value of $\tan\theta$.

We will first find the absolute value of $\tan\theta$. Then we will figure out the sign of $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{1-\cos^2\theta}}{\cos\theta} = \pm\frac{\sqrt{1-x^2}}{x}$$

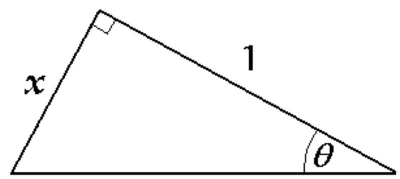
Let us consider the sign of $\tan\theta$ next. Because θ is in the range of $f(x) = \cos^{-1} x$, θ is in $[0, \pi]$. If $x = \cos\theta$ is negative, then θ is in the second quadrant. Then $\tan\theta$ is also negative. If $x = \cos\theta$ is positive, then θ is in the first quadrant, and then $\tan\theta$ is also positive. Considering the expression $\frac{\sqrt{1-x^2}}{x}$, we find that the numerator is positive for all values of x, and the denominator is positive if x is positive and negative if x is negative. This is exactly the sign we need and so $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

d) $\cos\left(\tan^{-1} x\right)$

Solution 1 (Geometric approach) We will first introduce a new variable. Let $\theta = \tan^{-1} x$. This means that θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = x$. (This also means that $-1 \le x \le 1$). We are asked to find the exact value of $\cos\theta$.

First we will first find the absolute value of $\cos\theta$. Then we will figure out the sign of $\cos\theta$.

For the absolute value of $\cos\theta$, let us assume that x is positive and think of it as $\frac{x}{1}$. $\tan\theta = x = \frac{x}{1}$. We draw a triangle where this is possible. We use the Pythagorean Theorem to find the missing side: $\sqrt{x^2+1}$. Thus $|\cos\theta| = \frac{1}{\sqrt{x^2+1}}$, or $\cos\theta = \pm\frac{1}{\sqrt{x^2+1}}$.



Let us consider the sign of $\cos\theta$ next. Because θ is in the range of $f(x) = \tan^{-1} x$, θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $x = \tan\theta$ is negative, then θ is in the fourth quadrant. If $x = \tan\theta$ is positive, then θ is in the first quadrant. In either case, $\cos\theta$ is positive and so the answer is $\cos\theta = \frac{1}{\sqrt{x^2+1}}$.

Solution 2 (Algebraic approach) We will first introduce a new variable. Let $\theta = \tan^{-1} x$. This means that θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = x$. (This also means that $-1 \le x \le 1$). We are asked to find the exact value of $\cos\theta$.

First we will first find the absolute value of $\cos\theta$. Then we will figure out the sign of $\cos\theta$.

Recall that $\tan^2\theta + 1 = \sec^2\theta$. (We can easily derive this by starting with $\sin^2\theta + \cos^2\theta = 1$ and divide both sides by $\cos^2\theta$).

$$\begin{aligned}
\sec^2\theta &= 1 + \tan^2\theta \\
\frac{1}{\cos^2\theta} &= 1 + \tan^2\theta \\
\cos^2\theta &= \frac{1}{1+\tan^2\theta} \\
\cos\theta &= \pm\sqrt{\frac{1}{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

Let us consider the sign of $\cos\theta$ next. Because θ is in the range of $f(x) = \tan^{-1} x$, θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $x = \tan\theta$ is negative, then θ is in the fourth quadrant. If $x = \tan\theta$ is positive, then θ is in the first quadrant. In either case, $\cos\theta$ is positive and so the answer is $\cos\theta = \frac{1}{\sqrt{x^2+1}}$.

6. Compute the exact value for each of the following.

a) $\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$

Solution: Let us first introduce a new variable. Let $\alpha = \sin^{-1}\left(\frac{4}{5}\right)$. We need to compute $\sin 2\alpha$.

Since α is in the range of $f(x) = \sin^{-1} x$, α is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because $\sin\alpha$ is positive, α is in the interval $\left[0, \frac{\pi}{2}\right]$ - thus $\cos\alpha$ is positive. Thus

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Now that we know that $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$, we can compute $\sin 2\alpha$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$$

b) $\cos\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)$

Solution: Let us first introduce a new variable. Let $\beta = \cos^{-1}\left(\frac{1}{3}\right)$. We need to compute $\cos 2\beta$.

$$\cos 2\beta = 2\cos^2\beta - 1 = 2\left(\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$$

c) $\tan\left(2\tan^{-1}\left(\frac{2}{5}\right)\right)$

Solution: Let us first introduce a new variable. Let $\theta = \tan^{-1}\left(\frac{2}{5}\right)$. We need to compute $\tan 2\theta$.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\frac{2}{5}}{1-\left(\frac{2}{5}\right)^2} = \frac{\frac{4}{5}}{1-\frac{4}{25}} = \frac{\frac{4}{5}}{\frac{21}{25}} = \frac{4}{5}\cdot\frac{25}{21} = \frac{20}{21}$$

d) $\sin\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$

Solution: Let us first introduce a new variable. Let $\alpha = \cos^{-1}\left(\frac{3}{4}\right)$. We need to compute $\sin 2\beta$.

Since α is in the range of $f(x) = \cos^{-1} x$, α is in the interval $[0, \pi]$. Because $\cos\alpha$ is positive, α is in the interval $\left[0, \frac{\pi}{2}\right]$ - thus $\sin\alpha$ is positive. Thus

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{4}\right)^2} = \sqrt{1-\frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

$$\sin 2\beta = 2\sin\beta\cos\beta = 2\cdot\frac{\sqrt{7}}{4}\cdot\frac{3}{4} = \frac{6\sqrt{7}}{16}$$

e) $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(-\frac{5}{13}\right)\right)$

Solution: Let us first introduce new variables. Let $\alpha=\sin^{-1}\left(\frac{3}{5}\right)$ and $\beta=\cos^{-1}\left(-\frac{5}{13}\right)$. We need to compute $\sin(\alpha+\beta)$.

Since α is in the range of $f(x)=\sin^{-1}x$, α is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because $\sin\alpha$ is positive, α is in the interval $\left[0,\frac{\pi}{2}\right]$ - thus $\cos\alpha$ is positive. Thus

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

Since β is in the range of $f(x)=\cos^{-1}x$, β is in the interval $[0,\pi]$. Because $\cos\beta$ is negative, β is in the interval $\left[\frac{\pi}{2},\pi\right]$ - thus $\sin\beta$ is positive. Thus

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(-\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

We now have all that we need: $\sin\alpha=\frac{3}{5}$, $\cos\alpha=\frac{4}{5}$, $\sin\beta=\frac{12}{13}$, $\cos\beta=-\frac{5}{13}$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{3}{5}\cdot\left(-\frac{5}{13}\right)+\frac{4}{5}\cdot\frac{12}{13}=\frac{-15}{65}+\frac{48}{65}=\frac{33}{65}$$

f) $\cos\left(\sin^{-1}\left(-\frac{1}{3}\right)-\cos^{-1}\left(\frac{2}{3}\right)\right)$

Solution: Let us first introduce new variables. Let $\alpha=\sin^{-1}\left(-\frac{1}{3}\right)$ and $\beta=\cos^{-1}\left(\frac{2}{3}\right)$. We need to compute $\cos(\alpha-\beta)$.

Since α is in the range of $f(x)=\sin^{-1}x$, α is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because $\sin\alpha$ is negative, α is in the interval $\left[-\frac{\pi}{2},0\right]$ - thus $\cos\alpha$ is positive. Thus

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(-\frac{1}{3}\right)^2}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{\sqrt{8}}{3}$$

Since β is in the range of $f(x)=\cos^{-1}x$, β is in the interval $[0,\pi]$. Because $\cos\beta$ is positive, β is in the interval $\left[0,\frac{\pi}{2}\right]$ - thus $\sin\beta$ is positive. Thus

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{2}{3}\right)^2}=\sqrt{1-\frac{4}{9}}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$$

We now have all that we need: $\sin\alpha=-\frac{1}{3}$, $\cos\alpha=\frac{\sqrt{8}}{3}$, $\sin\beta=\frac{\sqrt{5}}{3}$, $\cos\beta=\frac{2}{3}$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\frac{\sqrt{8}}{3}\cdot\frac{2}{3}+\left(-\frac{1}{3}\right)\cdot\frac{\sqrt{5}}{3}=\frac{2\sqrt{8}-\sqrt{5}}{9}=\frac{4\sqrt{2}-\sqrt{5}}{9}$$

g) $\tan\left(\tan^{-1}(3)+\tan^{-1}(7)\right)$

Solution: Let us first introduce new variables. Let $\alpha=\tan^{-1}(3)$ and $\beta=\tan^{-1}(7)$. Then $\tan\alpha=3$, $\tan\beta=7$, and we need to compute $\tan(\alpha+\beta)$.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{3+7}{1-3\cdot 7}=\frac{10}{1-21}=\frac{10}{-20}=-\frac{1}{2}$$

h) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{3}{4}\right)\right)$

Solution: Let us first introduce new variables. Let $\alpha=\sin^{-1}\left(\frac{1}{2}\right)$ and $\beta=\sin^{-1}\left(\frac{3}{4}\right)$. We need to compute $\sin(\alpha-\beta)$.

Since α is in the range of $f(x)=\sin^{-1}x$, α is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because $\sin\alpha$ is positive, α is in the interval $\left[0,\frac{\pi}{2}\right]$ - thus $\cos\alpha$ is positive. Thus

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{1}{2}\right)^2}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

Since β is in the range of $f(x)=\sin^{-1}x$, β is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because $\sin\beta$ is positive, β is in the interval $\left[0,\frac{\pi}{2}\right]$ - thus $\cos\beta$ is positive. Thus

$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\left(\frac{3}{4}\right)^2}=\sqrt{1-\frac{9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\frac{1}{2}\cdot\frac{\sqrt{7}}{4}-\frac{\sqrt{3}}{2}\cdot\frac{3}{4}=\frac{\sqrt{7}}{8}-\frac{3\sqrt{3}}{8}=\frac{\sqrt{7}-3\sqrt{3}}{8}$$