# Sample Problems

1. Prove that the area of a triangle can be computed as $A = \frac{1}{2} ab \sin \gamma$.

2. Prove the Law of Sines using results from problem 1.

3. Solve each of the following triangles. Present a decimal approximation of all answers, accurate up to four or more decimal places.

   a) $a = 3$, $\alpha = 42°$, $\beta = 100°$

   b) $a = 10$, $b = 12$, $\alpha = 32°$

   c) $a = 7 \quad b = 9 \quad \alpha = 114°$

   d) $a = 22\,\text{cm}$, $b = 23\,\text{cm}$, $\alpha = 67°$

   e) $a = 22\,\text{cm}$, $b = 23\,\text{cm}$, $\beta = 67°$

   f) $a = 6$, $b = 12$, $\alpha = 30°$

4. How many triangles are there with the data given?

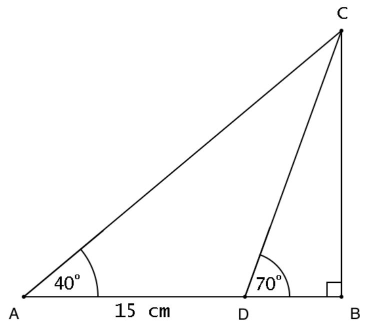
   a) $\alpha = 30°$, $b = 8$, $a = 3$

   b) $\alpha = 30°$, $b = 8$, $a = 4$

   c) $\alpha = 30°$, $b = 8$, $a = 5$

   d) $\alpha = 30°$, $b = 8$, $a = 9$

5. Compute the length of line segment $BD$ based on the picture given.




# Practice Problems

1. Solve each of the following triangles.

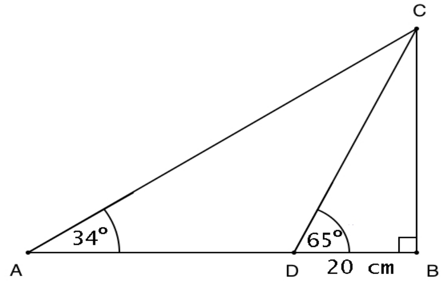
   a) $\alpha = 106°$, $\beta = 21°$, $b = 2.4\,\text{ft}$

   b) $a = 12\,\text{cm}$, $c = 17\,\text{cm}$, $\gamma = 85°$

   c) $a = 12\,\text{m}$, $b = 7\,\text{m}$, $\beta = 65°$

   d) $a = 15\,\text{ft}$, $c = 13\,\text{ft}$, $\gamma = 27°$

2. Compute the length of line segment $AC$ based on the picture given.



# Answers for Sample Problems

1. see solutions
2. see solutions
3. a) $\gamma = 38°$, $b \approx 4.41532$ unit, $c \approx 2.760275$ unit

   b) $\beta_1 = 39.487°$, $\gamma_1 = 108.513°$, $c_1 = 17.894$ and $\beta_2 = 140.513°$, $\gamma_2 = 7.487°$, $c_2 = 2.459$

   c) There is no triangle with the data given.

   d) $\beta_1 \approx 74.227°$, $\gamma_1 \approx 38.7730°$, $c_1 \approx 14.9670$ cm and $\beta_2 \approx 105.773°$, $\gamma_2 \approx 7.2270°$, $c_2 \approx 3.0066$ cm

   e) $\alpha \approx 61.70067°$, $\gamma \approx 51.29933°$, and $c \approx 19.499877$ cm.

   f) $\beta = 90°$, $\gamma = 60°$, and $c = 6\sqrt{3} \approx 10.392305$
4. a) no triangle b) one triangle c) two triangles d) one triangle
5. $6.59539$ cm

# Answers for Practice Problems

1. a) $\gamma = 53°$, $a \approx 6.4376$ ft, $c \approx 5.348484$ ft

   b) $\alpha \approx 44.684°$, $\beta \approx 50.316°$, $b \approx 13.1328$ cm

   c) no solution

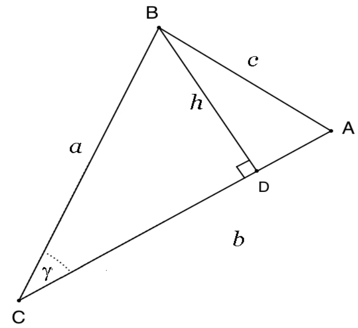
   d) $\alpha_1 = 31.43782584°$, $\beta_1 = 121.56217416°$, $b_1 \approx 24.399$ ft
   $\alpha_2 = 148.56217416°$, $\beta_2 = 4.43782584°$, $b_2 \approx 2.2157$ ft
2. $76.7$ cm

# Solutions for Sample Problems

1. Prove that the area of a triangle can be computed as $A = \frac{1}{2}ab\sin\gamma$.

   Proof: Consider the picture shown below.




   Let $h$ denote the height belonging to side $b$. The area of the triangle can be computed as $\frac{1}{2}bh$. Consider now the right triangle $BCD$. In this triangle, $\sin\gamma = \frac{h}{a}$. We solve for $h$: $h = a\sin\gamma$.

$$A = \frac{1}{2}bh = \frac{1}{2}b\,(a\sin\gamma) = \frac{1}{2}ab\sin\gamma$$

2. Prove the Law of Sines using results from problem 1.

   Proof: Consider triangle $ABC$. We compute the area of the triangle using the same formula but applying it to different sides.

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

   We now state just one such equality and cancel out a few things.

$$\begin{aligned} \frac{1}{2}ab\sin\gamma &= \frac{1}{2}ac\sin\beta && \text{divide by } \frac{1}{2}a \\ b\sin\gamma &= c\sin\beta && \text{divide by } bc \\ \frac{\sin\gamma}{c} &= \frac{\sin\beta}{b} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}ac\sin\beta &= \frac{1}{2}bc\sin\alpha && \text{divide by } \frac{1}{2}c \\ a\sin\beta &= b\sin\alpha && \text{divide by } ab \\ \frac{\sin\beta}{b} &= \frac{\sin\alpha}{a} \end{aligned}$$

   Thus we have proved that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

3. Solve each of the following triangles. Present a decimal approximation of all answers, accurate up to four or more decimal places.

a) $a = 3$, $\alpha = 42°$, $\beta = 100°$

Solution: Since two angles were given, we can easily compute the third.

$$\alpha + \beta + \gamma = 180° \implies \gamma = 180° - (\alpha + \beta) = 180° - (42° + 100°) = 38°$$

Now that we know all angles and one side, the Law of Sines will enable us to find the length of all sides. To compute $b$, we state the Law of Sines for $a$, $b$, and $\alpha$ and $\beta$ and solve for $b$. To make the algebra easier, we will chose a form of the theorem where the unknown, $b$ is in the numerator.

$$\frac{b}{\sin \beta} = \frac{a}{\sin \alpha} \implies b = \frac{a \sin \beta}{\sin \alpha} = \frac{3 \sin 100°}{\sin 42°} \approx 4.415\,316\,28$$

We find $c$ similarly: we state the Law of Sines for $a$, $c$, and $\alpha$ and $\gamma$ and solve for $c$.

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha} \implies c = \frac{a \sin \gamma}{\sin \alpha} = \frac{3 \sin 38°}{\sin 42°} \approx 2.760\,275$$

To solve a triangle means finding all sides and angles that were not given. Thus the answer is: $\gamma = 38°$, $b \approx 4.415\,316\,28$ unit, and $c \approx 2.760\,275$ unit.

b) $a = 10$, $b = 12$, $\alpha = 32°$

Solution: The only thing we can compute from this data is $\sin \beta$. We will do exactly that: we state the Law of Sines between $a$, $b$, $\alpha$ and $\beta$ and solve for $\sin \beta$.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \implies \sin \beta = \frac{b \sin \alpha}{a} = \frac{12 \sin 32°}{10} \approx 0.635\,903$$

Next we will solve for $\beta$. Recall that for all angles $\theta$, $\sin \theta = \sin (180° - \theta)$ and so there are two possible solutions between $0°$ and $180°$. When we enter $\sin^{-1}(0.635\,903)$ into the calculator, we obtain only one of the possible solutions. The other one is the complement of the first one.

$$\sin \beta = 0.635\,903 \implies \begin{cases} \beta_1 \approx 39.486\,999° \\ \beta_2 \approx 140.513\,001° \end{cases}$$

Sometimes both values work. From now on, we perform every step twice.

Now that we know $\beta$, we can compute the third angle, $\gamma$.

$$\begin{aligned} \gamma_1 &= 180° - (\alpha + \beta_1) = 180° - (32° + 39.486\,999°) = 108.513\,001° \\ \gamma_2 &= 180° - (\alpha + \beta_2) = 180° - (32° + 140.513\,001°) = 7.486\,999° \end{aligned}$$

Finally, we wil compute the length of side $c$. We will use the Law of Sines again.

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha} \implies c = \frac{a \sin \gamma}{\sin \alpha}$$

The two possible values of $c$ are then

$$\begin{aligned} c_1 &= \frac{a \sin \gamma_1}{\sin \alpha} = \frac{10 \sin 108.513\,001°}{\sin 32°} \approx 17.894\,266 \\ c_2 &= \frac{a \sin \gamma_2}{\sin \alpha} = \frac{10 \sin 7.486\,999°}{\sin 32°} \approx 2.458\,888 \end{aligned}$$

There are two solutions:

$\beta_1 = 39.487°$, $\gamma_1 = 108.513°$, $c_1 = 17.894$ and $\beta_2 = 140.513°$, $\gamma_2 = 7.487°$, $c_2 = 2.459$

c) $a = 7 \quad b = 9 \quad \alpha = 114°$

Solution: If we carefully look at the data given, we might see right away that there is something wrong. Because $a$ is less than $b$, we also have that $\alpha$ is less than $\beta$. But $\alpha = 114°$ and $\beta$ even greater is impossible. Let us see how the Law of Sines works. Because $a$, $b$, and $\alpha$ are given, we can solve for $\sin\beta$.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{9\sin 114°}{7} \approx 1.174\,56$$

The equation $\sin\beta \approx 1.174\,56$ has no solution because the sine of any angle is less than or equal to 1. This result is telling us that there is no triangle with the data given above.

d) $a = 22\,\text{cm}$, $b = 23\,\text{cm}$, $\alpha = 67°$

Solution: We can solve first for $\sin\beta$ by using the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{23\,\text{cm}\sin 67°}{22\,\text{cm}} \approx 0.962\,346$$

The equation $\sin\beta = 0.962346$ has two solutions between $0°$ and $180°$, and they are complements.

$$\sin\beta = 0.962\,346 \quad \Longrightarrow \quad \begin{cases} \beta_1 \approx 74.\,226955° \\ \beta_2 \approx 105.\,773\,045° \end{cases}$$

Let us now compute $\gamma$.

$$\begin{aligned} \gamma_1 &= 180° - (\alpha + \beta_1) = 180° - (67° + 74.\,226955°) = 38.\,773\,045° \\ \gamma_2 &= 180° - (\alpha + \beta_2) = 180° - (67° + 105.\,773\,045°) = 7.\,226\,955° \end{aligned}$$

In this case both values work. From now on, we perform every step twice. We can now compute $c$ via the Law of Sines.

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \quad \Longrightarrow \quad c = \frac{a\sin\gamma}{\sin\alpha}$$

The two possible values of $c$ are then

$$\begin{aligned} c_1 &= \frac{a\sin\gamma_1}{\sin\alpha} = \frac{22\,\text{cm}\sin 38.\,773045°}{\sin 67°} \approx 14.\,96702\,\text{cm} \\ c_2 &= \frac{a\sin\gamma_2}{\sin\alpha} = \frac{22\,\text{cm}\sin 7.\,226\,955°}{\sin 67°} \approx 3.\,00661\,\text{cm} \end{aligned}$$

Thus the two solutions are

$\beta_1 \approx 74.\,227°$, $\gamma_1 \approx 38.\,773\,0°$, $c_1 \approx 14.\,9670\,\text{cm}$ and $\beta_2 \approx 105.\,773°$, $\gamma_2 \approx 7.\,2270°$, $c_2 \approx 3.\,0066\,\text{cm}$

e) $a = 22\,\text{cm}$, $b = 23\,\text{cm}$, $\beta = 67°$

Solution: We can solve first for $\sin\alpha$ by using the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\alpha = \frac{a\sin\beta}{b} = \frac{22\,\text{cm}\sin 67°}{23\,\text{cm}} \approx 0.880\,482\,9$$

The equation $\sin\alpha = 0.880482\,9$ has two solutions between $0°$ and $180°$, and they are complements.

$$\sin\alpha = 0.880\,482\,9 \quad \Longrightarrow \quad \begin{cases} \alpha_1 \approx 61.\,700\,67° \\ \alpha_2 \approx 118.\,299\,33° \end{cases}$$

Let us now compute $\gamma$.

$$\begin{aligned} \gamma_1 &= 180° - (\alpha_1 + \beta) = 180° - (67° + 61.\,700\,67°) = 51.\,299\,33° \\ \gamma_2 &= 180° - (\alpha_2 + \beta) = 180° - (67° + 118.\,299\,33°) = -5.\,299\,33° \end{aligned}$$

Since the sum of $\beta$ and $\alpha_2$ is greater than $180°$, the numbers here are telling us that there is only one triangle with the data given. So, $\gamma \approx 51.29933°$. We can now compute $c$ via the Law of Sines.

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \quad \Longrightarrow \quad c = \frac{b\sin\gamma}{\sin\beta} = \frac{23\,\text{cm}\sin 51.29933°}{\sin 67°} \approx 19.499877\,\text{cm}$$

Thus the solution is $\alpha \approx 61.70067°$, $\gamma \approx 51.29933°$, and $c \approx 19.499877\,\text{cm}$.

f) $a = 6$, $b = 12$, $\alpha = 30°$

Solution: We can solve first for $\sin\beta$ by using the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{12\sin 30°}{6} = 1$$

The equation $\sin\beta = 1$ has only one solution between $0°$ and $180°$, it is $\beta = 90°$. Let us now compute $\gamma$.

$$\gamma = 180° - (\alpha+\beta) = 180° - (30° + 90°) = 60°$$

In this case the triangle has a right angle and so we can compute $c$ by applying the Pythagorean theorem. Since $\beta = 90°$, the hypotenuse is $b$.

$$c = \sqrt{b^2 - a^2} = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$$

Thus the solution is $\beta = 90°$, $\gamma = 60°$, and $c = 6\sqrt{3} \approx 10.392305$

4. How many triangles are there with the data given?

a) $\alpha = 30°$, $b = 8$, $a = 3$

Solution: We can first solve for $\sin\beta$ via the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{8\sin 30°}{3} = \frac{4}{3}$$

Since $\frac{4}{3} > 1$, the equation $\sin\beta = \frac{4}{3}$ has no solution and so there is no triangle with the data given.

b) $\alpha = 30°$, $b = 8$, $a = 4$

Solution: We can first solve for $\sin\beta$ via the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{8\sin 30°}{4} = 1$$

The equation $\sin\beta = 1$ has one solution between $0°$ and $180°$ so there is one triangle with the data given.

c) $\alpha = 30°$, $b = 8$, $a = 5$

Solution: We can first solve for $\sin\beta$ via the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{8\sin 30°}{5} = \frac{4}{5}$$

The equation $\sin\beta = \frac{4}{5}$ has two solutions between $0°$ and $180°$ so there might be two triangles with the data given.

$$\sin\beta = \frac{4}{5} \quad \Longrightarrow \quad \begin{cases} \beta_1 \approx 53.1301° \\ \beta_2 \approx 126.8699° \end{cases}$$

We need to compute $\gamma$ to see whether both values work.

$$\begin{aligned} \gamma_1 &= 180° - (\alpha + \beta_1) = 180° - (30° + 53.1301°) = 96.8699° \\ \gamma_2 &= 180° - (\alpha + \beta_2) = 180° - (30° + 126.8699°) = 23.1301° \end{aligned}$$

At this point we know that there are two triangles with the data given.

d) $\alpha = 30°, \ b = 8, \ a = 9$

Solution: We can first solve for $\sin\beta$ via the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{8\sin 30°}{9} = \frac{4}{9}$$

The equation $\sin\beta = \dfrac{4}{9}$ has two solutions between 0° and 180° so there might be two triangles with the data given.

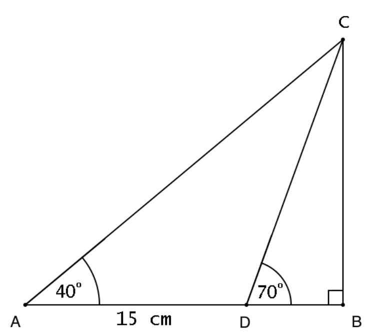
$$\sin\beta = \frac{4}{9} \quad \Longrightarrow \quad \begin{cases} \beta_1 \approx 26.3878° \\ \beta_2 \approx 153.6122° \end{cases}$$

We need to compute $\gamma$ to see whether both values work.

$$\begin{aligned} \gamma_1 &= 180° - (\alpha + \beta_1) = 180° - (30° + 26.3878°) = 123.6122° \\ \gamma_2 &= 180° - (\alpha + \beta_2) = 180° - (30° + 153.6122°) = -3.6122° \end{aligned}$$

At this point we know that there is only one triangle with the data given.

5. Compute the length of line segment $BD$ based on the picture below.




Solution: Angle $ADC = 110°$. Now we can compute every side and angle in triangle $ADC$ using the Law of Sines.

$$\measuredangle ACD = 180° - (40° + 110°) = 30°$$

We will now compute the length of line segment $CD$.

$$\frac{CD}{\sin 40°} = \frac{15\,\text{cm}}{\sin 30°} \quad \Longrightarrow \quad CD = \frac{15\,\text{cm}\sin 40°}{\sin 30°} \approx 19.2836283\,\text{cm}$$

We will now use right triangle trigonometry to compute the length of line segment $BD$.

$$\cos 70° = \frac{BD}{CD} \quad \Longrightarrow \quad BD = CD\cos 70° \approx 19.2836283\,\text{cm}\cos 70° \approx 6.59539\,\text{cm}$$

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