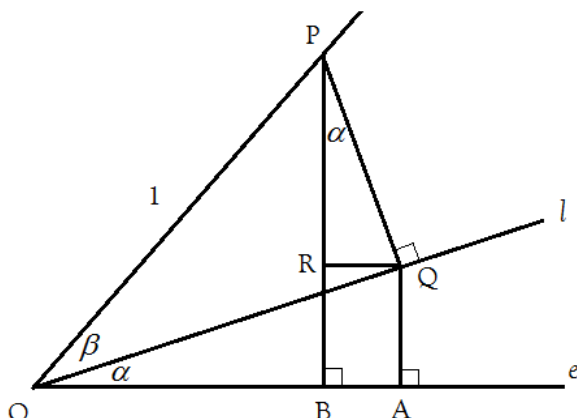


Consider the picture below.



Let O be any point from which we measure the angles α and β as shown. Let P be at a distance of 1 unit to O . We draw two lines passing through P that are perpendicular to rays e and l , respectively. The intersection points created are Q and B . From Q , we draw a line that is perpendicular to BP , the intersection point is R . Finally, we draw a line passing through Q that is perpendicular to ray e . The intersection point is A .

Claim 1: $\angle QPR = \alpha$

proof: First notice that $\angle RQO = \alpha$ because OB and RQ are parallel. Second, $\angle RQO$ and $\angle RQP$ are complementary angles, they add up to 90° .

$$\begin{aligned}\angle RQO + \angle RQP &= 90^\circ \\ \alpha + \angle RQP &= 90^\circ \\ \angle RQP &= 90^\circ - \alpha\end{aligned}$$

Now the angles in triangle PQR must add to 180° :

$$\begin{aligned}\angle QPR + 90^\circ + \angle RQP &= 180^\circ \\ \angle QPR + 90^\circ + 90^\circ - \alpha &= 180^\circ \\ \angle QPR - \alpha &= 0 \\ \angle QPR &= \alpha\end{aligned}$$

Claim 2: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

proof: Consider the right triangle OPQ .

$$\sin \beta = \frac{PQ}{OP} = \frac{PQ}{1} = PQ \implies \boxed{PQ = \sin \beta} \quad \text{and} \quad \cos \beta = \frac{OQ}{OP} = \frac{OQ}{1} = OQ \implies \boxed{OQ = \cos \beta}$$

Consider now the right triangle OAQ : $\sin \alpha = \frac{AQ}{OQ} = \frac{AQ}{\cos \beta} \implies AQ = \sin \alpha \cos \beta$

From right triangle PRQ , $\cos \alpha = \frac{PR}{PQ} = \frac{PR}{\sin \beta} \implies PR = \cos \alpha \sin \beta$

From right triangle OBP ,

$$\sin(\alpha + \beta) = \frac{BP}{OP} = \frac{BP}{1} = BP = PR + RB = PR + AQ = \cos \alpha \sin \beta + \sin \alpha \cos \beta$$

Claim 3: $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

proof: Consider the right triangle OPQ . $OP = 1$ and $PQ = \sin \beta$ and $OQ = \cos \beta$

From right triangle OAQ , $\cos \alpha = \frac{AO}{OQ} = \frac{AO}{\cos \beta} \implies AO = \cos \alpha \cos \beta$

From right triangle PQR , $\sin \alpha = \frac{QR}{PQ} = \frac{QR}{\sin \beta} \implies QR = \sin \alpha \sin \beta$

From right triangle OBP ,

$$\cos(\alpha + \beta) = \frac{OB}{OP} = \frac{AO - QR}{1} = AO - QR = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

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