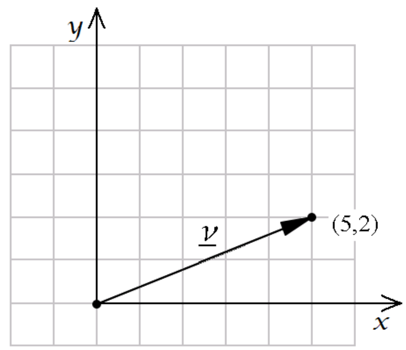
Vectors Represented Algebraically

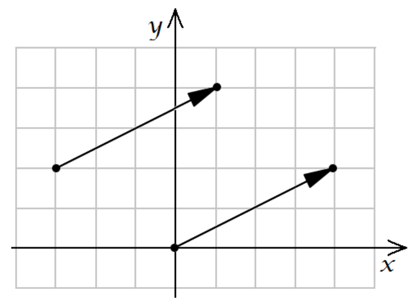
Definition: A position vector $\underline{v}$ is represented as $\underline{v} = \langle a, b \rangle$ is the vector with initial point $(0,0)$ and terminal point (a,b). The scalars a and b are called the components of $\underline{v}$.



$\underline{v} = \langle 5, 2 \rangle$

Equality of Vectors: If $\underline{v} = \langle a, b \rangle$ and $\underline{w} = \langle c, d \rangle$, then $\underline{v} = \underline{w}$ if and only if $a = c$ and $b = d$.

We can write every vector as a position vector. For example, the vector pointing from $A(-3,2)$ to $B(1,4)$ can be represented as the position vector $\langle 4, 2 \rangle$.



The magnitude of a vector $\underline{v} = \langle a, b \rangle$, denoted by $\|\underline{v}\|$, can be computed as

$$\|\underline{v}\| = \sqrt{a^2 + b^2}$$

If $\|\underline{v}\| = 1$, the vector is called a unit vector.

In our example, $\|\underline{v}\| = \sqrt{5^2 + 2^2} = \sqrt{29}$

Another, very important representation of vectors is by the use of the horizontal and vertical unit vectors. If $\langle 1, 0 \rangle = \underline{i}$ and $\langle 0, 1 \rangle = \underline{j}$, then every vector can be expressed as a linear combination of $\underline{i}$ and $\underline{j}$. For example, $\langle 5, 2 \rangle = 5\underline{i} + 2\underline{j}$.

$$\underline{v} = \langle a, b \rangle = a\underline{i} + b\underline{j}$$

a and b are also called the horizontal and vertical components of $\underline{v}$.

Properties of $\|\underline{v}\|$

If $\underline{v}$ is a vector and α is a scalar, then

a) $\|\underline{v}\| \geq 0$

b) $\|\underline{v}\| = 0$ if and only if $\underline{v} = \underline{0}$.

c) $\|-\underline{v}\| = \|\underline{v}\|$

d) $\|\alpha \underline{v}\| = |\alpha| \, \|\underline{v}\|$

Operations on Vectors

Suppose that $\underline{v} = \langle a, b\rangle$ and $\underline{w} = \langle c, d\rangle$ are vectors and α is a scalar. The sum and difference of $\underline{v}$ and $\underline{w}$ are defined as

$$\begin{aligned}\underline{v} + \underline{w} &= \langle a+c, b+d\rangle = (a+c)\,\underline{i} + (b+d)\,\underline{j}\\ \underline{v} - \underline{w} &= \langle a-c, b-d\rangle = (a-c)\,\underline{i} + (b-d)\,\underline{j}\\ \alpha\underline{v} &= \langle \alpha a, \alpha b\rangle = (\alpha a)\,\underline{i} + (\alpha b)\,\underline{j}\end{aligned}$$

Example: Suppose that $\underline{v} = \langle -2, 3\rangle$ and $\underline{w} = \langle 1, 4\rangle$. Compute each of the following.

a) $\underline{v} + \underline{w}$ b) $\underline{v} - \underline{w}$ c) $3\underline{v} - 4\underline{w}$

Solution: a) $\underline{v} + \underline{w} = \left(-2\underline{i} + 3\underline{j}\right) + \left(\underline{i} + 4\underline{j}\right) = -2\underline{i} + 3\underline{j} + \underline{i} + 4\underline{j} = \left(-2\underline{i} + \underline{i}\right) + \left(3\underline{j} + 4\underline{j}\right) = -\underline{i} + 7\underline{j}$

b) $\underline{v} - \underline{w} = \left(-2\underline{i} + 3\underline{j}\right) - \left(\underline{i} + 4\underline{j}\right) = -2\underline{i} + 3\underline{j} - \underline{i} - 4\underline{j} = \left(-2\underline{i} - \underline{i}\right) + \left(3\underline{j} - 4\underline{j}\right) = -3\underline{i} - \underline{j}$

c) $3\underline{v} - 4\underline{w} = 3\left(-2\underline{i} + 3\underline{j}\right) - 4\left(\underline{i} + 4\underline{j}\right) = -6\underline{i} + 9\underline{j} - 4\underline{i} - 16\underline{j} = (-6-4)\,\underline{i} + (9-16)\,\underline{j} = -10\underline{i} - 7\underline{j}$

Different Representations of Vectors

Sometimes it is useful to give the magnitude and the direction of a vector. For example, let $\underline{v}$ be a vector with magnitude 5 units and direction $120°$ with the positive part of the $x-$axis. To express this vector using horizontal and vertical components.

$$\begin{aligned}\text{horizontal component} &= 5\cos 120° = -2.5\\ \text{vertical component} &= 5\sin 120° = 5\cdot\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}\end{aligned}$$

$$\text{So}\quad \underline{v} = -2.5\underline{i} + \frac{5\sqrt{3}}{2}\underline{j}$$

If a vector is given via horizontal and vertical components, we could also express it by stating the magnitude and the direction of a vector. For example, let $\underline{w} = 4\underline{i} - \underline{j}$.

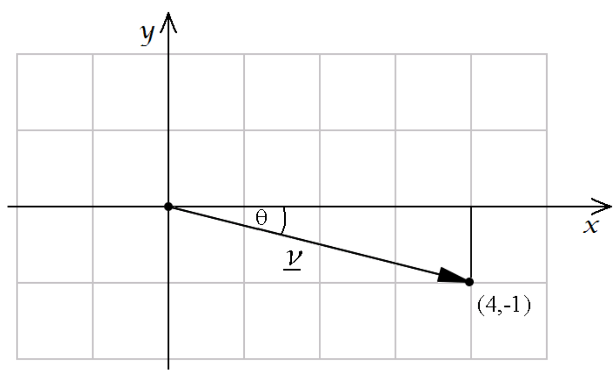
The magnitude is then

$$\|\underline{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

For the direction, we observe that the terminal point is in the fourth quadrant. Then using right triangle trigonometry,

$$\theta = \tan^{-1}\left(\frac{1}{4}\right) \approx 14.036\,243°$$

and so the direction of the vector is $-14.036\,243°$ with respect to the positive part of the $x-$axis.



Finding a unit vector

Suppose we need to replace a vector $\underline{v} = 5\underline{i} - 2\underline{j}$ with a unit vector with the same direction. What we need to do then is divide the vector by its own magnitude. In this case,

$$\frac{\underline{v}}{\|\underline{v}\|} = \frac{5\underline{i} - 2\underline{j}}{\sqrt{(5)^2 + (-2)^2}} = \frac{5\underline{i} - 2\underline{j}}{\sqrt{29}} = \frac{5}{\sqrt{29}}\underline{i} - \frac{2}{\sqrt{29}}\underline{j}$$

This vector is a unit vector since $\sqrt{\left(\frac{5}{\sqrt{29}}\right)^2 + \left(-\frac{2}{\sqrt{29}}\right)^2} = \sqrt{\frac{25}{29} + \frac{4}{29}} = 1$

Practice Problems

1. Find an algebraic expression for each of the vectors given.
   a) the vector pointing from $A(3, -2)$ to $B(-2, 9)$.
   b) the vector pointing from $B(-2, 9)$ to $A(3, -2)$.

2. Given the vectors $\underline{u} = 2\underline{i} - \underline{j}$, $\underline{v} = -5\underline{i}$ and $\underline{w} = \underline{i} + \underline{j}$, compute each of the following.

   a) $\underline{u} + \underline{v}$
   b) $\underline{v} - \underline{w}$
   c) $2\underline{u} - \underline{v} + \underline{w}$
   d) $3\underline{w} - 2\underline{v}$
   e) $5\underline{u} + 3\underline{v} + 5\underline{w}$
   f) $-2\underline{u} + 3\underline{v} - \underline{w}$

3. Given the vectors $\underline{a} = -\underline{i} + 3\underline{j}$, $\underline{b} = \underline{i} - 2\underline{j}$, and $\underline{c} = -7\underline{i} + \underline{j}$, compute each of the following.

   a) $\underline{a} + \underline{b}$
   b) $3\underline{b}$
   c) $\underline{b} - \underline{a}$
   d) $\underline{a} + \underline{b} + \underline{c}$
   e) $\underline{a} - 2\underline{b} + \underline{c}$
   f) $-2\underline{c} = -2c$
   g) $-2\underline{a} + 3\underline{b} - \underline{c}$

4. Given the vectors $\underline{a} = -\underline{i} + 3\underline{j}$, $\underline{b} = \underline{i} - 2\underline{j}$, and $\underline{c} = -7\underline{i} + \underline{j}$, find real numbers A, B, and C, not all zero, so that
$$A\underline{a} + B\underline{b} + C\underline{c} = \underline{0}$$

5. Find the length and direction for each of the following vectors given.
   a) $\underline{a} = -\underline{i} + 3\underline{j}$ b) $\underline{b} = 4\underline{i} + 4\underline{j}$ c) $\underline{c} = -7\underline{i} - 3\underline{j}$ d) $\underline{d} = 2\underline{i} - \underline{j}$

6. Express the given vectors using horizontal and vertical components.

   a) magnitude: 10 direction: 30°
   b) magnitude: 6 direction: 135°
   c) magnitude: 7 direction: 270°
   d) magnitude: 2 direction: 85°

7. Find a unit vector with the same direction as the vector given.
   a) $\underline{v} = -3\underline{i} + 4\underline{j}$ b) $\underline{w} = -\underline{i} + \underline{j}$ c) $\underline{u} = 4\underline{i} - 7\underline{j}$

Answers - Practice Problems

1. a) $\underline{v} = \langle -5, 11 \rangle = -5\underline{i} + 11\underline{j}$ b) $\underline{w} = \langle 5, -11 \rangle = 5\underline{i} - 11\underline{j}$
2. a) $-3\underline{i} - \underline{j}$ b) $-6\underline{i} - \underline{j}$ c) $10\underline{i} - \underline{j}$ d) $13\underline{i} + 3\underline{j}$ e) $\underline{0}$ f) $-20\underline{i} + \underline{j}$
3. a) $\underline{j}$ b) $3\underline{i} - 6\underline{j}$ c) $2\underline{i} - 5\underline{j}$ d) $-7\underline{i} + 2\underline{j}$ e) $-10\underline{i} + 8\underline{j}$ f) $14\underline{i} - 2\underline{j}$ g) $12\underline{i} - 13\underline{j}$
4. Answers might vary $A = 13$, $B = 20$, $C = 1$ works.
5. a) magnitude: $\sqrt{10}$ direction: $\pi - \tan^{-1}(3) \approx 108.434\,949°$
   b) magnitude: $4\sqrt{2}$ direction: $\dfrac{\pi}{4} = 45°$
   c) magnitude: $\sqrt{58}$ direction: $\pi + \tan^{-1}\left(\dfrac{3}{7}\right) \approx 203.198\,5905°$
   d) magnitude: $\sqrt{5}$ direction: $-\tan^{-1}\left(\dfrac{1}{2}\right) \approx -26.565\,0512°$
6. a) $5\underline{i} + 5\sqrt{3}\underline{j}$ b) $3\sqrt{2}\underline{i} - 3\sqrt{2}\underline{j}$ c) $-7\underline{j}$ d) $0.174\,3115\underline{i} + 1.992\,4\underline{j}$
7. a) $-\dfrac{3}{5}\underline{i} + \dfrac{4}{5}\underline{j}$ b) $-\dfrac{\sqrt{2}}{2}\underline{i} + \dfrac{\sqrt{2}}{2}\underline{j}$ c) $\dfrac{4}{\sqrt{65}}\underline{i} - \dfrac{7}{\sqrt{65}}\underline{j}$

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