The Dot Product

Definition: The dot product (or scalar product) of vectors is defined as follows. If two vectors are perpendicular, their dot product is zero. If two vectors are parallel, their dot product is the product of their magnitudes. In addition, the dot product is commutative, and the distributive law also holds, i.e for all vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$.

Form this definition follows that $\underline{i} \cdot \underline{i} = 1$, $\underline{j} \cdot \underline{j} = 1$, and $\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i} = 0$

Suppose that $\underline{u} = a\underline{i} + b\underline{j}$ and $\underline{v} = c\underline{i} + d\underline{j}$

$$\begin{aligned} \underline{u} \cdot \underline{v} &= \left(a\underline{i} + b\underline{j}\right) \cdot \left(c\underline{i} + d\underline{j}\right) = ac\underline{i} \cdot \underline{i} + ad\underline{i} \cdot \underline{j} + bc\underline{j} \cdot \underline{i} + bd\underline{j} \cdot \underline{j} \\ &= ac \cdot 1 + ad \cdot 0 + bc \cdot 0 + bd \cdot 1 = ac + bd \end{aligned}$$

Theorem: If $\underline{u} = a\underline{i} + b\underline{j}$ and $\underline{v} = c\underline{i} + d\underline{j}$, then $\underline{u} \cdot \underline{v} = ac + bd$.

There is another way to think of the dot product. Suppose that $\underline{u}$ and $\underline{v}$ are vectors. We can decompose $\underline{v}$ into two components: one that is parallel to $\underline{u}$ and one that is perpendicular to $\underline{u}$. Thus $\underline{v} = \underline{v}_{\text{par}} + \underline{v}_{\text{perp}}$.

$$\text{Then} \quad \begin{aligned} \underline{u} \cdot \underline{v} &= \underline{u} \cdot \left(\underline{v}_{\text{par}} + \underline{v}_{\text{perp}}\right) = \underline{u} \cdot \underline{v}_{\text{par}} + \underline{u} \cdot \underline{v}_{\text{perp}} = \|\underline{u}\| \left\|\underline{v}_{\text{par}}\right\| + 0 \\ &= \|\underline{u}\| \left\|\underline{v}_{\text{par}}\right\| \end{aligned}$$

The parallel component $\underline{v}_{\text{par}}$ has length $\|\underline{v}\| \cos\theta$ where θ is the angle formed between $\underline{u}$ and $\underline{v}$. Thus

If $\underline{u}$ and $\underline{v}$ are vectors and θ is the angle formed between them, then

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$$

The fact that the dot product can be computed in two such different ways enables us to compute the angle formed between two vectors. Suppose that $\underline{u} = 3\underline{i} - \underline{j}$ and $\underline{v} = 4\underline{i} + 5\underline{j}$. We will use the dot product to compute the angle formed between $\underline{u}$ and $\underline{v}$. The dot product is

$$\underline{u} \cdot \underline{v} = \left(3\underline{i} - \underline{j}\right) \cdot \left(4\underline{i} + 5\underline{j}\right) = 3 \cdot 4 + (-1) \cdot 5 = 12 - 5 = 7$$

$$\text{On the other hand,} \quad \begin{aligned} \|\underline{u}\| &= \sqrt{3^2 + (-1)^2} = \sqrt{10} \quad \text{and} \quad \|\underline{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41} \\ \underline{u} \cdot \underline{v} &= \|\underline{u}\| \|\underline{v}\| \cos\theta = \sqrt{10}\sqrt{41} \cos\theta \end{aligned}$$

The dot product is the same, no matter how we compute it. So,

$$\begin{aligned} \sqrt{10}\sqrt{41} \cos\theta &= 7 \\ \cos\theta &= \frac{7}{\sqrt{410}} \\ \theta &= \cos^{-1}\left(\frac{7}{\sqrt{410}}\right) \approx 69.775\,14^\circ \end{aligned}$$

Practice Problems

1. Compute the dot product of the pairs of vectors given.

   a) $\underline{u} = 2\underline{i} - \underline{j}$ and $\underline{v} = -5\underline{i} + 2\underline{j}$

   b) $\underline{u} = 5\underline{i} - \underline{j}$ and $\underline{v} = \underline{i} + 5\underline{j}$

   c) $\underline{u} = -2\underline{i} - 5\underline{j}$ and $\underline{v} = 4\underline{i} + 3\underline{j}$

2. Compute the angle between the pairs of vectors given. Present the exact value and an approximation.

   a) $\underline{u} = 5\underline{i} - 4\underline{j}$ and $\underline{v} = 2\underline{i} + 2\underline{j}$

   b) $\underline{u} = 3\underline{i} + 4\underline{j}$ and $\underline{v} = -\underline{i} + 7\underline{j}$

   c) $\underline{u} = 4\underline{i} - \underline{j}$ and $\underline{v} = 2\underline{i} + 8\underline{j}$ $^\circ$

   d) $\underline{u} = -2\underline{i} - 4\underline{j}$ and $\underline{v} = 7\underline{i} + \underline{j}$

Answers - Practice Problems

1. a) -12 b) 0 c) -23

2. a) $\cos^{-1}\left(\frac{\sqrt{82}}{82}\right) \approx 83.65981°$ b) $45°$ c) $90°$ d) $\cos^{-1}\left(-\frac{9\sqrt{10}}{50}\right) \approx 124.695154°$