Calculus is a study of functions. We usually ask many questions about the function. Together, this is called the complete analysis of the function. The following is just a part of the complete list, which will grow as we advance in our studies.

Equation: $f(x) =$ ________________

domain: ________________

range: ________________

$y-$intercept: ________________

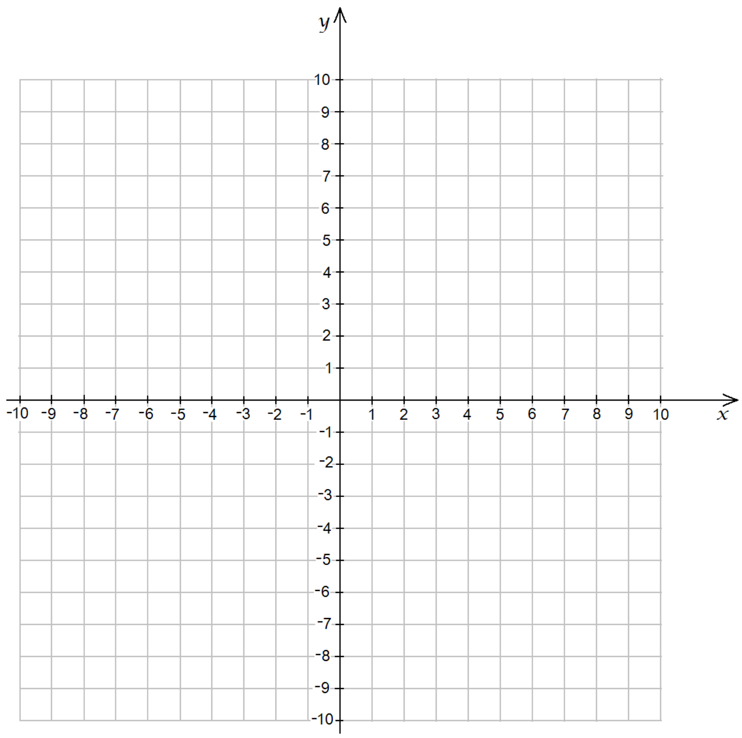
$x-$intercept(s): ________________

maximum: ________________

minimum: ________________

one-to-one: ________________

graph:



Definitions

1. The **domain** of a function is a non-empty set of elements to which we assign things.

   The **range** of a function is the set of elements that we assign to elements of the domain.

2. The $y-$**intercept** of a function is the point where the graph of the function intersects the $y-$axis. A function can have at most one $y-$intercept.

   The $x-$**intercept** of a function is the point where the graph of the function intersects the $x-$axis. A function can have several $x-$intercepts.

3. Extremum is a common name for a maximum or a minimum.

   A function f has an **absolute maximum** at x_M if for all x in the domain, $f(x_M) \geq f(x)$.

   A function f has an **absolute minimum** at x_m if for all x in the domain, $f(x_m) \leq f(x)$.

4. Definition: A function f is **one-to-one (or injective)** if for all a and b in its domain, if $a \neq b$, then $f(a) \neq f(b)$.

   Alternative definition: A function f is **one-to-one (or injective)** if for all a and b in its domain, if $f(a) = f(b)$, then $a = b$.

Sample Problems

Sketch the graph and give a complete analysis for each of the following functions.

1. a) $f(x) = -\frac{2}{3}x + 4$

   b) $f(x) = -\frac{2}{3}x + 4$ on domain $[-3, 9]$

2. a) $f(x) = x^2 - 6x + 5$

   b) $f(x) = x^2 - 6x + 5$ on domain $[-1, 6]$

3. a) $f(x) = -16x^2 + 64x + 192$

   b) $f(x) = -16x^2 + 64x + 192$ on domain $[3, 7]$

Sample Problems - Solutions

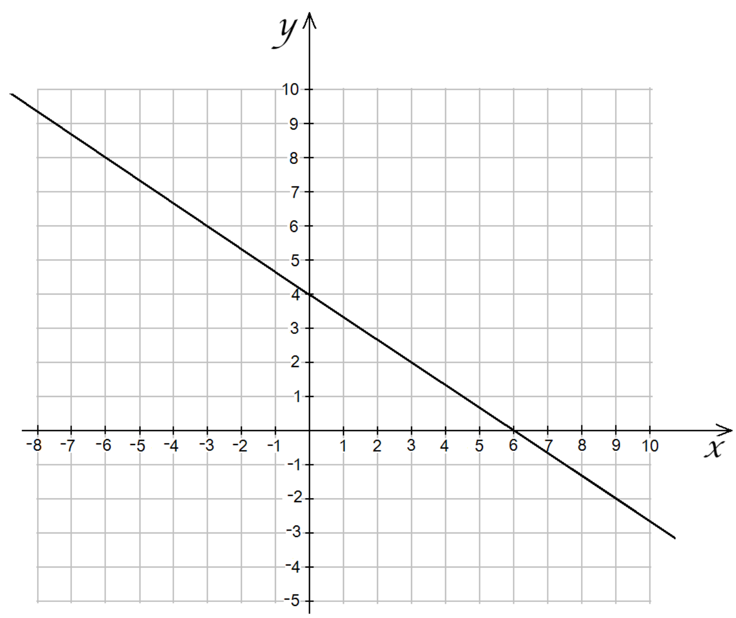
Sketch the graph and give a complete analysis for each of the following functions.

1. a) $f(x) = -\dfrac{2}{3}x + 4$

Solution: to compute the $y-$intercept, we substitute $x = 0$ and easily obtain $f(0) = 4$. For the $x-$intercept, we write $y = -\dfrac{2}{3}x + 4$ and solve for x in $y = 0$

$$\begin{aligned} y &= -\frac{2}{3}x + 4 && x = ? \text{ so that } y = 0 \\ 0 &= -\frac{2}{3}x + 4 && \text{add } \frac{2}{3}x \\ \frac{2}{3}x &= 4 && \text{divide by } \frac{2}{3} \\ x &= \frac{4}{\frac{2}{3}} = 4 \cdot \frac{3}{2} = 6 \end{aligned}$$

Lines are uniquely determined by two points. We can already graph this function using its intercepts: $(0, 4)$ and $(6, 0)$.



The analysis of the funtion can be summarized:

domain: $\mathbb{R}$	$y-$intercept: $(0, 4)$	maximum: there is none
range: $\mathbb{R}$	$x-$intercept: $(6, 0)$	minimum: there is none
	one-to-one: yes	graph: see above

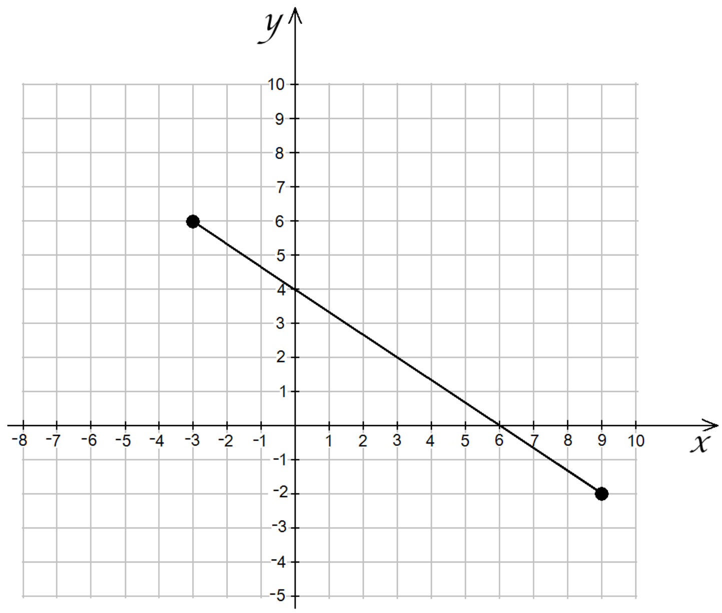
b) $f(x) = -\frac{2}{3}x + 4$ on domain $[-3, 9]$

This function has the same assignment as in the previous problem, only the domain is different. We will see in our study of calculus that the domain is just as important factor in the behavior of a function as is its assignment.

First we perform the same computations as before and sketch the same graph as before. In addition, we need to evaluate the function at $x = -3$ and $x = 9$.

$$f(-3) = -\frac{2}{3}(-3) + 4 = 6 \quad \text{and} \quad f(9) = -\frac{2}{3}(9) + 4 = -2$$

So now our graph is adjusted based on this new domain:



The analysis of the funtion is changed as the domain is different.

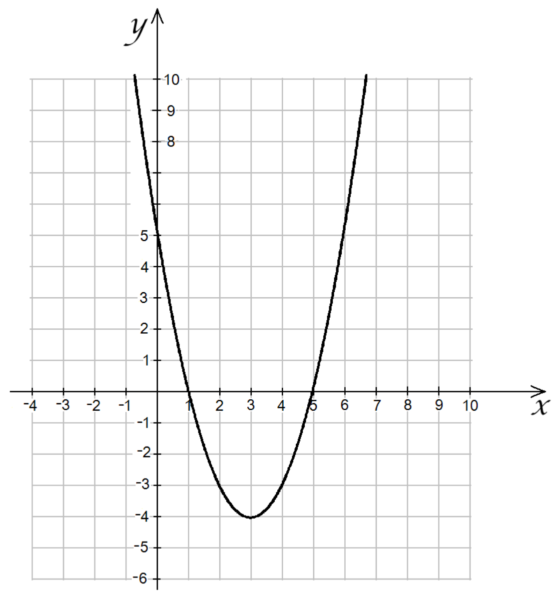
domain: $[-3, 9]$	$y-$intercept: $(0, 4)$	maximum: $(-3, 6)$
range: $[-2, 6]$	$x-$intercept: $(6, 0)$	minimum: $(9, -2)$
	one-to-one: yes	graph: see above

2. a) $f(x) = x^2 - 6x + 5$

Solution: We start with some algebra:

$$\begin{aligned} y &= x^2 - 6x + 5 \\ &= x^2 - 6x + 9 - 9 + 5 \\ &= (x-3)^2 - 4 \\ &= (x-3+2)(x-3-2) \\ &= (x-1)(x-5) \end{aligned} \qquad (x-3)^2 = x^2 - 6x + 9$$

This computation tells us a lot of useful things. The graph of this function is an upward opening parabola, the $y-$intercept is $(0,5)$, the $x-$intercepts are $(1,0)$ and $(5,0)$, and the vertex is $(3,-4)$. Based on this, we graph f.



When we graph a function, we already know all we wanted to know about it. The domain is the set of all real numbers (or $\mathbb{R}$) because any real number can be substituted into the formula of the function. We have found the vertex to be $(3,-4)$ and the parabola opens upward: so the range is all numbers greater than or equal to -4. In interval notation: $[-4,\infty)$. This function has no maximum but there is a minimum: $(3,-4)$. Finally, this function is clearly not one-to-one because $y-$values are taken by several $x-$values. For example, $f(2) = -3$ and $f(4) = -3$, or just the fact that there are two $x-$intercepts means that the function assigns zero to two different $x-$values. We summarize all this:

domain: $\mathbb{R}$	$y-$intercept: $(0,5)$	maximum: there is none
range: $[-4,\infty)$	$x-$intercepts: $(1,0)$ and $(5,0)$	minimum: $(3,-4)$
	one-to-one: no	graph: see above

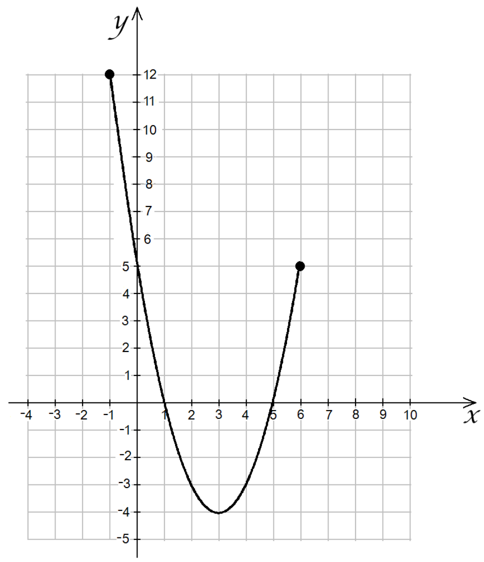
b) $f(x) = x^2 - 6x + 5$ on domain $[-1, 6]$

This function has the same assignment as in the previous problem, only the domain is different. We will see in our study of calculus that the domain is just as important factor in the behavior of a function as is its assignment.

First we perform the same computations as before and sketch the same graph as before. In addition, we need to evaluate the function at $x = -1$ and $x = 6$.

$$f(-1) = (-1)^2 - 6(-1) + 5 = 12 \quad \text{and} \quad f(6) = 6^2 - 6 \cdot 6 + 5 = 5$$

So now our graph is adjusted based on this new domain:



The new domain changes quite a few things. This function has both a maximum and a minimum, and the range is much smaller:

domain: $[-1, 6]$
range: $[-4, 12]$

$y-$intercept: $(0, 5)$
$x-$intercepts: $(1, 0)$ and $(5, 0)$
one-to-one: no

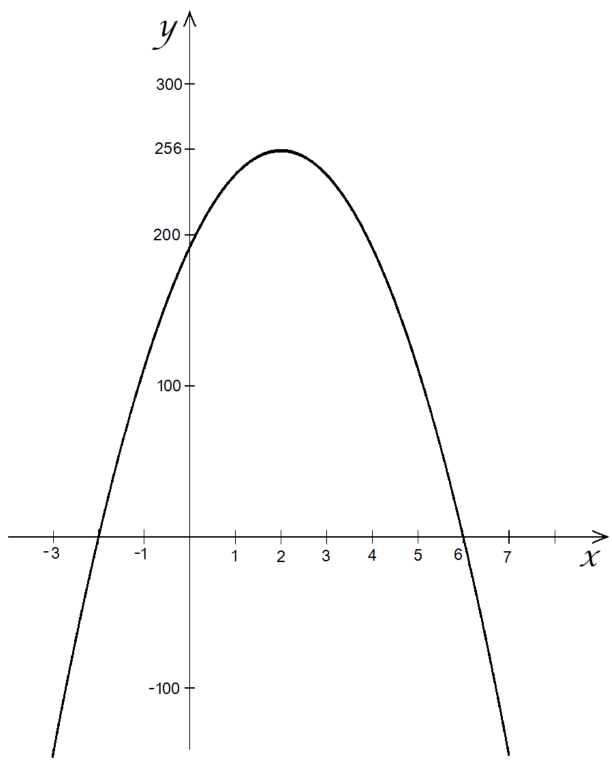
maximum: $(-1, 12)$
minimum: $(3, -4)$
graph: see above

3. a) $f(x) = -16x^2 + 64x + 192$

Solution: We start with some algebra:

$$\begin{aligned} y &= -16x^2 + 64x + 192 \qquad \text{factor out } -16 \\ &= -16\left(x^2 - 4x - 12\right) \qquad (x-2)^2 = x^2 - 4x + 4 \\ &= -16\left(x^2 - 4x + 4 - 4 - 12\right) \\ &= -16\left[(x-2)^2 - 16\right] = -16(x-2)^2 + 256 \\ &= -16(x - 2 + 4)(x - 2 - 4) \\ &= -16(x+2)(x-6) \end{aligned}$$

This computation tells us many useful things. The graph of this function is a downward opening parabola, the y-intercept is $(0, 192)$, the x-intercepts are $(-2, 0)$ and $(6, 0)$, and the vertex is $(2, 256)$. Based on this, we can now graph f.



The complete analysis of this graph is:

domain: $\mathbb{R}$
range: $(-\infty, 256]$

y-intercept: $(0, 192)$
x-intercepts: $(-2, 0)$ and $(6, 0)$
one-to-one: no

maximum: $(2, 256)$
minimum: there is none
graph: see above

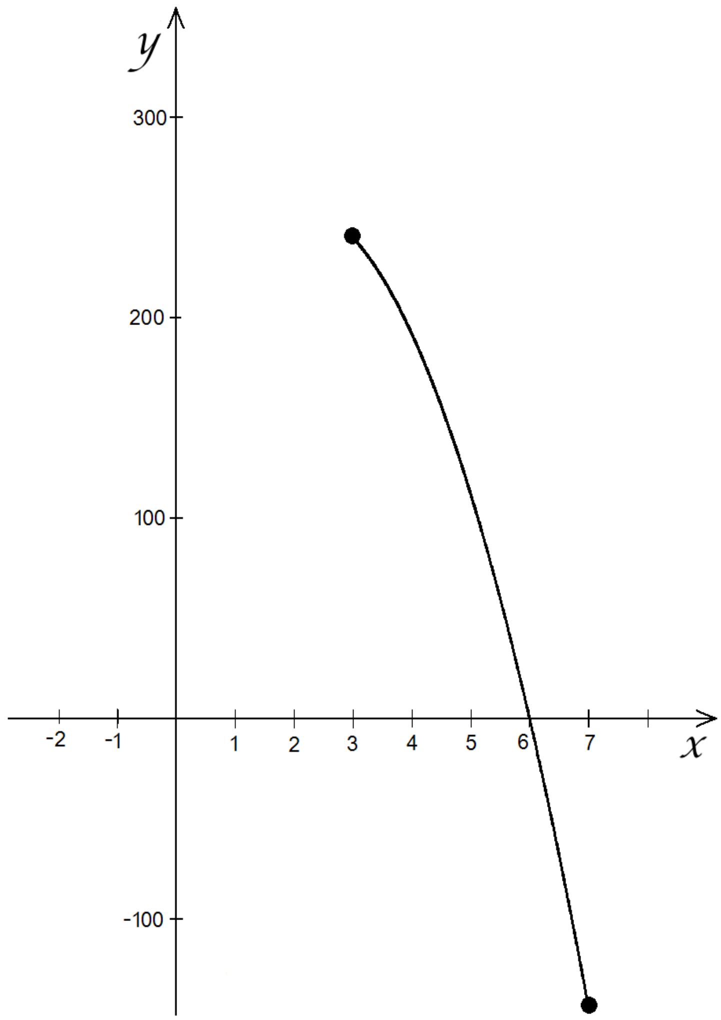
b) $f(x) = -16x^2 + 64x + 192$ on domain $[3, 7]$

This function has the same assignment as in the previous probem, only the domain is different. We will see in our study of calculus that the domain is just as important factor in the behavior of a function as is its assignment.

First we perform the same computations as before and sketch the same graph as before. In addition, we need to evaluate the function at $x = 3$ and $x = 7$.

$$\begin{aligned} f(3) &= -16 \cdot 3^2 + 64 \cdot 3 + 192 = -16 \cdot 9 + 192 + 192 = -144 + 192 + 192 = 240 \text{ and} \\ f(7) &= -16 \cdot 7^2 + 64 \cdot 7 + 192 = -16 \cdot 49 + 64 \cdot 7 + 192 = -784 + 448 + 192 = -144 \end{aligned}$$

Also notice that the vertex of this parabola is at $x = 2$, but the new domain does not include it, only the numbers between 3 and 7. Our graph is adjusted based on this new domain:



The new domain changes quite a few things. This function has both a maximum and a minimum, and the range is much smaller. Also, this function is one-to-one.

domain: $[3, 7]$
range: $[-144, 240]$
$y-$intercept: none
$x-$intercept: $(6, 0)$
one-to-one: yes
maximum: $(3, 240)$
minimum: $(7, -144)$
graph: see above

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