The following is the actual definition of a (finite) limit of a function f at a number c.

Definition: Suppose that f is a function and c, L are real numbers. We say that $\lim_{x\to c} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \neq c$ with $|x - c| < \delta$, we also have that (f is defined and) $|f(x) - L| < \varepsilon$.

In this course, we will not use the rigorous definition, we will use instead the following.

Definition: If the left-hand side limit and the right-hand side limit both exist (and are finite) and are equal, we say that $\lim_{x\to c} f(x) = L$.

Continuity is an extremely important property of functions that will have significant impact on other behaviors of functions. A function is continuous at a point if there is a two-sided, finite limit, and it is also the function value.

Definition: A function $y = f(x)$ is **continuous at a number** c of its domain if the two-sided limit exists, $f(c)$ exists, and $\lim_{x\to c} f(x) = f(c)$.

Continuity as deifned above is a local property, defined point by point. However, it will be beneficial to also define continuity on an interval.

Definition: (Continuity on an open interval). Suppose that I is an open interval, i.e. $I = (a, b)$ or $I = (-\infty, b)$ or $I = (a, \infty)$. A function $y = f(x)$ is continuous on I if it is continuous at every real number c that lies in I.

We will also define continuity on a closed interval $[a, b]$, as continuity on a closed interval will turn out to have extremely nice properties.

Definition: (Continuity on a closed interval) A function $y = f(x)$ is continuous on $[a, b]$ if

1) f is continuous at every number c within the interval (a, b). (A number c is sometimes called an interior point of the interval).
2) f is right-continuous at $x = a$, i.e. $\lim_{x\to a^+} f(x)$ exists, $f(a)$ exists, and $f(a) = \lim_{x\to a^+} f(x)$
3) f is left-continuous at $x = b$, i.e. $\lim_{x\to b^-} f(x)$ exists, $f(b)$ exists, and $f(b) = \lim_{x\to b^-} f(x)$.

Also note that another way to express continuity is to say that $\lim_{h\to 0} f(x + h) = f(x)$. Another alternative statement of continuity is $\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right)$, so that there is a commutativity between taking the limit and taking the function values.

Theorem: Suppose that f and g are functions that are continuous at $x = c$. Then:

1) $f + g$ is continuous at c.
2) fg is continuous at c
3) $f - g$ is continuous at c
4) If $g(c) \neq 0$, then $\dfrac{f}{g}$ is continuous at c
5) If g is continuos at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

These properties can be proved using the properties of limits and the definitions of the functions $f + g$, fg, $f - g$, $\dfrac{f}{g}$ and $f \circ g$.

Example 1: Suppose that f is a function defined as $f(x) = \begin{cases} mx - 10 & \text{if } x < -2 \\ x^2 + 9x - 8 & \text{if } x \geq -2 \end{cases}$. Find the value of m if we know that f is continuous everywhere.

Solution: If $x < -2$, then the function is continuous for all x. Similarly, f is also continuous on all x with $x \geq -2$. The only questionable point is at $x = -2$. For a continuous function, we need the left limit and the right limit to exist and have the same value.

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^+} f(x) \\ \lim_{x \to -2^-} (mx - 10) &= \lim_{x \to -2^+} \left(x^2 + 9x - 8\right) \end{aligned}$$

By the various properties of limits, this equation can be simplified as follows:

$$\begin{aligned} m(-2) - 10 &= (-2)^2 + 9(-2) - 8 \\ -2m - 10 &= -22 \\ -2m &= -12 \\ m &= 6 \end{aligned}$$

And so $m = 6$ is the value for which f is continuous on the entire number line.

Practice Problems

1. Suppose that f is a function defined as $f(x) = \begin{cases} mx - 13 & \text{if } x < -10 \\ x^2 + 5x - 3 & \text{if } x \geq -10 \end{cases}$. Find the value of m if we know that f is continuous everywhere.
2. Suppose that f is a function defined as $f(x) = \begin{cases} 8x - 4 & \text{if } x \leq 4 \\ -2x + b & \text{if } x > 4 \end{cases}$. Find the value of b if we know that f is continuous everywhere.
3. Suppose that f is a function defined as $f(x) = \begin{cases} mx - 11 & \text{if } x < -6 \\ x^2 + 4x - 5 & \text{if } x \geq -6 \end{cases}$. Find the value of m if we know that f is continuous everywhere.
4. Suppose that f is a function defined as $f(x) = \begin{cases} 2x + b & \text{if } x < 7 \\ \sqrt{x + 2} & \text{if } x \geq 7 \end{cases}$. Find the value of b if we know that f is continuous everywhere.

Answers - Practice Problems

1.) -6 2.) 36 3.) -3 4) -11

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