Sample Problems

1. Find an equation for the tangent line drawn to graph of f at the given value of x.

   a) $f(x) = \sqrt{x}$ at $x = 9$ b) $f(x) = x^4 - 2x^3 + 3$ at $x = -1$

2. Let $f(x) = 16x + \dfrac{1}{x}$. Find the equation of all tangent lines drawn to the graph of f that are perpendicular to the line $x + 12y = -5$.

3. Find the values of a and b if we know that $y = -5x + 20$ is a tangent line drawn to the graph of $f(x) = ax^3 + bx^2 - x + 8$ at $x = 2$.

4. Find all values of p for which the line $y = x$ is a tangent line to the graph of $f(x) = px^2 + 6$.

5. Find an equation for all tangent lines drawn to the graph of $f(x) = \dfrac{1}{2}x^2 + 3x - 2$ from the point $P(-1, -9)$.

Practice Problems

1. Find an equation for the tangent line drawn to graph of f at the given value of x.

   a) $f(x) = 3x^4 + 2x - 7$ at $x = -1$

   b) $f(x) = x^2 - 12\sqrt{x}$ at $x = 4$

   c) $f(x) = \dfrac{1}{x^3} - \dfrac{1}{x}$ at $x = 1$

   d) $f(x) = -x^3 + 5x^2 + x - 8$ at $x = 2$

   e) $f(x) = 2x^3 + x^2 - 5x - 3$ at $x = -2$

   f) $f(x) = x^3 + 3x^2 + 3x - 12$ at $x = -1$

   g) $f(x) = 3x^4 + 6x^3 - 9x^2 - 8x + 2$ at $x = -2$

   h) $f(x) = 3x^4 + 6x^3 - 9x^2 - 8x + 2$ at $x = 1$

2. a) Let $f(x) = x^2 - 5x + 1$. Find the equation of all tangent lines drawn to the graph of f that are parallel to the line $x + y = 15$.

   b) Let $f(x) = x^3 - 8x + 6$. Find an equation for all tangent lines drawn to the graph of f that are perpendicular to the line $x + 4y = -12$.

   c) Let $f(x) = x - 3\sqrt{x}$. Find an equation for all tangent lines drawn to the graph of f that are perpendicular to the line $2x + y = 10$.

3. a) Find the values of a and b if we know that $y = -6x + 19$ is a tangent line drawn to the graph of $f(x) = ax^2 + bx + 3$ at $x = 2$.

   b) Find the values of m and n if we know that $y = 8x - 1$ is a tangent line drawn to the graph of $f(x) = mx^3 + nx^2 + x - 5$ at $x = -1$.

   c) Find the values of p and q if we know that $y = -\dfrac{1}{2}x + 7$ is a tangent line drawn to the graph of $f(x) = p\sqrt{x} + qx + 1$ at $x = 4$.

4. a) Find all values of p for which the line $y = 2x + 4$ is a tangent line to the graph of $f(x) = px^2 + 10$.

   b) Find all values of m for which the line $y = mx + \dfrac{1}{2}$ is a tangent line to the graph of $f(x) = \sqrt{x} - 2$.

   c) Find all values of A for which the line $y = 13x + 18$ is a tangent line to the graph of $f(x) = Ax^2 - 3x + 2$.

   d) Find all values of T for which the line $y = -2x + 12$ is a tangent line to the graph of $f(x) = T\sqrt{x} - 5x$.

5. In each case, find an equation for all tangent lines drawn to the graph of $f(x)$ given from the point given.

a) $y = x^2 - 6x + 1$ from $P(3, -9)$

b) $y = \frac{1}{2}x^2 - 6x + 30$ from $P(7, 8)$

c) $y = x^2 - 2x - 8$ from $P(3, -14)$

d) $y = x^2 - 8x + 3$ from $P(0, -6)$

e) $f(x) = 7x - \frac{1}{2}x^2 - 20$ from $P(4, 8)$

f) $y = \frac{1}{2}x^2 + 3x - 11$ from $P(2, -5)$

g) $f(x) = \frac{1}{2}x^2 - x + 2$ from $P(3, -1)$

Sample Problems - Answers

1. a) $y = \frac{1}{6}x + \frac{3}{2}$ b) $y = -10x - 4$
2. $y = 12x + 4$ and $y = 12x - 4$.
3. $a = -2$ and $b = 5$
4. $m = \frac{1}{24}$
5. $y = -x - 10$ and $y = 5x - 4$

Practice Problems - Answers

1. a) $y = -10x - 16$ b) $y = 5x - 28$ c) $y = -2x + 2$ d) $y = 9x - 12$
   e) $y = 15x + 25$ f) $y = -13$ g) $y = 4x - 10$ h) $y = 4x - 10$
2. a) $y = -x - 3$ b) $y = 4x - 10$ and $y = 4x + 22$ c) $y = \frac{1}{2}x - \frac{9}{2}$
3. a) $a = -4$, $b = 10$ b) $m = 1, n = -2$ c) $p = 6, q = -2$
4. a) $p = \frac{1}{6}$ b) $m = \frac{1}{10}$ c) $A = -4$ d) $T = 12$
5. a) $y = 2x - 15$ and $y = -2x - 3$ b) $y = -2x + 22$ and $y = 4x - 20$ c) $y = -2x - 8$ and $y = 10x - 44$
   d) $y = -2x - 6$ and $y = -14x - 6$ e) $y = -x + 12$ and $y = 7x - 20$ f) $y = 3x - 11, y = 7x - 19$
   g) $y = -x + 2$ and $y = 5x - 16$

Sample Problems - Solutions

1. Find an equation for the tangent line drawn to graph of f at the given value of x.

a) $f(x) = \sqrt{x}$ at $x = 9$

Solution: For the equation of a line, we need a point and the line's slope. The point of tangency $(9, f(9))$ is on the line. Since $f(9) = \sqrt{9} = 3$, the point $(9, 3)$ is on the line. The derivative is the slope of the tangent line. So, the slope is the derivative, evaluated at $x = 9$, in short, $m = f'(9)$.

We differentiate $f(x)$.

$$f(x) = \sqrt{x} = x^{1/2}$$

By the power rule, $f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. We evaluate this at $x = 3$

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

So the tangent line has slope $\frac{1}{6}$ and passes through the point $(9, 3)$. We can immediately write the point-slope form of the equation and transform it to the slope-intercept form.

$$\begin{aligned} y - 3 &= \frac{1}{6}(x - 9) \\ y &= \frac{1}{6}(x - 9) + 3 = \frac{1}{6}x - \frac{9}{6} + 3 = \frac{1}{6}x - \frac{3}{2} + \frac{6}{2} \\ y &= \frac{1}{6}x + \frac{3}{2} \end{aligned}$$

b) $f(x) = x^4 - 2x^3 + 3$ at $x = -1$

Solution: For the equation of a line, we need a point and the line's slope. The point of tangency $(-1, f(-1))$ is on the line. Since $f(-1) = (-1)^4 - 2(-1)^3 + 3 = 1 + 2 + 3 = 6$, the point $(-1, 6)$ is on the line. The derivative is the slope of the tangent line. So, the slope is the derivative, evaluated at $x = -1$, in short, $m = f'(-1)$.

We differentiate $f(x)$.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 3 \\ f'(x) &= 4x^3 - 2\left(3x^2\right) + 0 = 4x^3 - 6x^2 \end{aligned}$$

$f'(x) = 4x^3 - 6x^2$. We evaluate this at $x = -1$

$$f'(-1) = 4(-1)^3 - 6(-1)^2 = -4 - 6 = -10$$

So the tangent line has slope -10 and passes through the point $(-1, 6)$. We can immediately write the point-slope form of the equation and transform it to the slope-intercept form.

$$\begin{aligned} y - 6 &= -10(x + 1) \\ y &= -10(x + 1) + 6 = -10x - 10 + 6 \\ y &= -10x - 4 \end{aligned}$$

2. Let $f(x) = 16x + \dfrac{1}{x}$. Find the equation of all tangent lines drawn to the graph of f that are perpendicular to the line $x + 12y = -5$.

Solution: First we compute the slope of the line $x + 12y = -5$ by solving for y.

$$\begin{aligned} x + 12y &= -5 \\ 12y &= -x - 5 \\ y &= \frac{-x-5}{12} = -\frac{1}{12}x - \frac{5}{12} \end{aligned}$$

Our line must be perpendicular to this line with slope $-\dfrac{1}{12}$. The slope of our line is therefore the negative reiprocal of $-\dfrac{1}{12}$, which is 12. So now the question is: where are the tangent lines drawn to f that have slopes 12? That is the same as looking for all values of f for which $f'(x) = 12$.

$$\begin{aligned} f(x) &= 16x + \frac{1}{x} = 16x + x^{-1} \\ f'(x) &= 16 + (-1)x^{-2} = 16 - \frac{1}{x^2} \\ 12 &= 16 - \frac{1}{x^2} && \text{add}\frac{1}{x^2} \\ \frac{1}{x^2} + 12 &= 16 && \text{subtract } 12 \\ \frac{1}{x^2} &= 4 \\ \frac{1}{x} &= \pm 2 \quad \Longrightarrow \quad x = \pm\frac{1}{2} \end{aligned}$$

Let us check: if $x = -\dfrac{1}{2}$, then $f'(x) = 16 - \left(\dfrac{1}{\left(-\frac{1}{2}\right)^2}\right) = 16 - 4 = 12$. Clearly $x = \dfrac{1}{2}$ also works. For the tangent line, we compute the point of tangency: $\left(\dfrac{1}{2}, f\left(\dfrac{1}{2}\right)\right)$ and $\left(-\dfrac{1}{2}, f\left(-\dfrac{1}{2}\right)\right)$.

$$f\left(\frac{1}{2}\right) = 16\left(\frac{1}{2}\right) + \frac{1}{\left(\frac{1}{2}\right)} = 8 + 2 = 10 \quad \text{and} \quad f\left(-\frac{1}{2}\right) = 16\left(-\frac{1}{2}\right) + \frac{1}{\left(-\frac{1}{2}\right)} = -8 - 2 = -10$$

So one tangent line has slope $m = 12$ and passes through $\left(\dfrac{1}{2}, 10\right)$ will have the equation

$$\begin{aligned} y - 10 &= 12\left(x - \frac{1}{2}\right) \\ y &= 12x - 6 + 10 \\ y &= 12x + 4 \end{aligned}$$

The other tangent line has slope $m = 12$ and passes through $\left(-\dfrac{1}{2}, -10\right)$ will have the equation

$$\begin{aligned} y + 10 &= 12\left(x + \frac{1}{2}\right) \\ y &= 12x + 6 - 10 \\ y &= 12x - 4 \end{aligned}$$

So the tangent lines we are looking for are $y = 12x + 4$ and $y = 12x - 4$.

3. Find the values of a and b if we know that $y = -5x + 20$ is a tangent line drawn to the graph of $f(x) = ax^3 + bx^2 - x + 8$ at $x = 2$.

Solution: The tangent line at $x = 2$. So it's slope, -5 is the derivative at $x = 2$. In short, $f'(2) = -5$. Also, the point of tangency is at $x = 2$

$$y = -5(2) + 20 = -10 + 20 = 10$$

and so both the line and the graph of f contains the point $(2, 10)$. So we now know that

$$f(2) = 10 \quad \text{and } f'(2) = -5$$

These will give us two equations

$$f(2) = 10 = a \cdot 2^3 + b \cdot 2^2 - 2 + 8 = 8a + 4b + 6$$

$$\begin{aligned} 10 &= 8a + 4b + 6 \\ 4 &= 8a + 4b \\ 1 &= 2a + b \end{aligned}$$

$$\begin{aligned} f(x) &= ax^3 + bx^2 - x + 8 \\ f'(x) &= 3ax^2 + 2bx - 1 \\ -5 &= 3a \cdot 2^2 + 2b \cdot 2 - 1 \\ -5 &= 12a + 4b - 1 \\ -4 &= 12a + 4b \\ -1 &= 3a + b \end{aligned}$$

Now we have two equations in the variables a and b.

$$\begin{aligned} 1 &= 2a + b \\ -1 &= 3a + b \end{aligned}$$

We will use elimination. We will multiply the first equation by -1 and add the two equations.

$$\begin{aligned} -1 &= -2a - b \\ -1 &= 3a + b \end{aligned}$$

We add the two equations and obtain $-2 = a$. We substitute this in the first equation and solve for b

$$\begin{aligned} 1 &= 2a + b \\ 1 &= 2(-2) + b \\ 1 &= -4 + b \\ 5 &= b \end{aligned}$$

So the solution is: $a = -2$ and $b = 5$ and so $f(x) = -2x^3 + 5x^2 - x + 8$.

4. Find all values of p for which the line $y = x$ is a tangent line to the graph of $f(x) = px^2 + 6$.

Solution: Let $P(a, f(a))$ be the point of tangency. First, P is on the graph of $f(x)$:

$$f(a) = pa^2 + 6$$

Second, P is also on the line $y = x$ and so $f(a) = a$.

$$a = pa^2 + 6$$

This is now one equation for two unknowns, p and a. We will obtain the second equation by stating that the derivative at $x = a$ is 1 (the slope of the tangent line).

$$\begin{aligned} f(x) &= px^2 + 6 \\ f'(x) &= 2px \end{aligned}$$

The derivative is 1 at $x = a$

$$\begin{aligned} 1 &= 2pa \\ \frac{1}{2a} &= p \end{aligned}$$

We substitute it into the other equation:

$$\begin{aligned} a &= pa^2 + 6 \quad \text{where} \quad \frac{1}{2a} = p \\ a &= \left(\frac{1}{2a}\right) a^2 + 6 \\ a &= \frac{a}{2} + 6 \\ a - \frac{a}{2} &= 6 \\ \frac{a}{2} &= 6 \\ a &= 12 \end{aligned}$$

This means that the point of tangency is $P(12, 12)$. We can also compute the value of $p = \frac{1}{2a} = \frac{1}{2(12)} = \frac{1}{24}$.

So our function is $f(x) = \frac{1}{24}x^2 + 6$. We check:

$$\begin{aligned} f(x) &= \frac{1}{24}x^2 + 6 & \qquad f(12) &= \frac{1}{24}(12)^2 + 6 = 6 + 6 = 12 \\ f'(x) &= \frac{1}{24}(2x) = \frac{1}{12}x & \qquad f'(12) &= \frac{1}{12}(12) = 1 \end{aligned}$$

So the tangent line drawn to f at $x = 12$ is

$$\begin{aligned} 1(x - 12) &= y - 12 \\ x - 12 &= y - 12 \\ x &= y \end{aligned}$$

And so our solution is correct.

5. Find the equation of all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 + 3x - 2$ from the point $P(-1,-9)$.

Solution: This problem is more difficult because the point given, P is not on the graph of f. Let $Q(a, f(a))$ be the point of tangency. First, Q is on the graph of f

$$f(a) = \frac{1}{2}a^2 + 3a - 2$$

Second, the derivative of f, evaluated at $x = a$ is the slope of the tangent line.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x + 3 \qquad \Longrightarrow \qquad f'(a) = a + 3 \end{aligned}$$

This is also the slope of the line determined by $P(-1,-9)$ and $Q\left(a, \frac{1}{2}a^2 + 3a - 2\right)$.

$$m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(a) - (-9)}{a - (-1)} = \frac{\frac{1}{2}a^2 + 3a - 2 + 9}{a + 1} = \frac{\frac{1}{2}a^2 + 3a + 7}{a + 1}$$

The slope of the tangent line is the derivative evaluated at $x = a$.

$$\begin{aligned} m &= f'(a) \\ a + 3 &= \frac{\frac{1}{2}a^2 + 3a + 7}{a + 1} && \text{multiply by } a + 1 \\ (a+3)(a+1) &= \frac{1}{2}a^2 + 3a + 7 \\ a^2 + 4a + 3 &= \frac{1}{2}a^2 + 3a + 7 && \text{subtract 7} \\ a^2 + 4a - 4 &= \frac{1}{2}a^2 + 3a && \text{multiply by 2} \\ 2a^2 + 8a - 8 &= a^2 + 6a \\ a^2 + 2a - 8 &= 0 \\ (a+4)(a-2) &= 0 \qquad \Longrightarrow \qquad a_1 = -4, \ a_2 = 2 \end{aligned}$$

Case 1. If $a = -4$. Then $f(a) = \frac{1}{2}(-4)^2 + 3(-4) - 2 = -6$ and so the point of tangency is $(-4,-6)$. The slope of the tangent line is $f'(-4)$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x + 3 \\ f'(-4) &= -1 \end{aligned}$$

and so the tangent line is $-1(x+4) = y + 6$ or $y = -x - 10$. This line indeed contains $P(-1,-9)$.

Case 2. If $a = 2$. Then $f(a) = \frac{1}{2}(2)^2 + 3(2) - 2 = 6$ and so the point of tangency is $(2,6)$. The slope of the tangent line is $f'(2)$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x + 3 \\ f'(2) &= 5 \end{aligned}$$

and so the tangent line is $5(x-2) = y - 6$ or $y = 5x - 4$. This line also contains $P(-1,-9)$.

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