The derivative of a function of a real variable expresses the rate of change of a quantity (a function value or dependent variable) which is determined by another quantity (the independent variable). Derivatives are a fundamental tool of calculus. For example, the derivative of the location function of a moving object with respect to time is the object's velocity: it measures how quickly the position of the object changes when time is advanced.

Suppose that f is a function. Let a be a fixed number in the domain of f.

The expression $\dfrac{f(a+h)-f(a)}{h}$ is called the **difference quotient**. The difference quotient has a geometric meaning: it is the slope of the secant line connecting two points on the graph of f: $A(a, f(a))$ and $B(a+h, f(a+h))$. The difference quotient also has an interpretation in physics: if the function f is location function, then the difference quotient expresses the average velocity between times $t_1 = a$ and $t_2 = a+h$.

The **derivative** of f, at the number a, denoted by $f'(a)$, is defined as the limit of the difference quotient:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

The derivative of f at a is only defined if the limit shown above exists and is finite.

The derivative also has a geometric meaning: $f'(a)$ is the slope of the tangent line drawn to the graph of f at pont $(a, f(a))$. The difference quotient also has an interpretation in physics: if the function f is location function, then $f'(a)$ expresses the instantaneous velocity at $t = a$.

Given a function f, if we evaluate $f'(x)$ for all x, we obtain a new function, called the derivative (or first derivative) of f.

Differentiate each of the following by evaluating the limit of the difference quotient.

1. $f(x) = x^2 - 3x$
2. $f(x) = x^3$
3. $f(x) = \sqrt{2x-1}$
4. $f(x) = \dfrac{1}{x^2-1}$
5. $f(x) = \sqrt{1-x^2}$

Solutions

1. $f(x) = x^2 - 3x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left((x+h)^2 - 3(x+h)\right) - \left(x^2-3x\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(x^2+2xh+h^2-3x-3h\right) - \left(x^2-3x\right)}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} \frac{h(2x+h-3)}{h} = \lim_{h\to 0}(2x+h-3) = 2x-3 \end{aligned}$$

2. $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2 \end{aligned}$$

3. $f(x) = \sqrt{2x-1}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{2x+2h-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2x+2h-1} + \sqrt{2x-1}}{\sqrt{2x+2h-1} + \sqrt{2x-1}}\right) \\ &= \lim_{h\to 0} \frac{(2x+2h-1)-(2x-1)}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)} = \lim_{h\to 0} \frac{2x+2h-1-2x+1}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)} \\ &= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h-1} + \sqrt{2x-1}} = \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}} \end{aligned}$$

4. $f(x) = \dfrac{1}{x^2-1}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2-1} - \dfrac{1}{x^2-1}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{x^2-1}{\left((x+h)^2-1\right)\left(x^2-1\right)} - \dfrac{(x+h)^2-1}{\left(x^2-1\right)\left((x+h)^2-1\right)}}{h} = \lim_{h\to 0} \left(\frac{x^2-1-\left((x+h)^2-1\right)}{\left((x+h)^2-1\right)\left(x^2-1\right)} \cdot \frac{1}{h}\right) \\ &= \lim_{h\to 0} \frac{x^2-1-\left(x^2+2xh+h^2-1\right)}{h\left((x+h)^2-1\right)\left(x^2-1\right)} = \lim_{h\to 0} \left(\frac{x^2-1-x^2-2xh-h^2+1}{h\left((x+h)^2-1\right)\left(x^2-1\right)}\right) \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{h\left((x+h)^2-1\right)\left(x^2-1\right)} = \lim_{h\to 0} \frac{-h(2x+h)}{h\left((x+h)^2-1\right)\left(x^2-1\right)} = \lim_{h\to 0} \frac{-(2x+h)}{\left((x+h)^2-1\right)\left(x^2-1\right)} \\ &= \frac{-2x}{\left(x^2-1\right)\left(x^2-1\right)} = \frac{-2x}{\left(x^2-1\right)^2} \end{aligned}$$

5. $f(x) = \sqrt{1-x^2}$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h} \cdot \frac{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}}{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}} \\
&= \lim_{h\to 0} \frac{1-(x+h)^2-\left(1-x^2\right)}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0} \frac{1-x^2-h^2-2xh-1+x^2}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0} \frac{-h^2-2xh}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0} \frac{-\cancel{h}(h+2x)}{\cancel{h}\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0} \frac{-(h+2x)}{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}} = \frac{-2x}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}
\end{aligned}$$

6. $f(x) = \dfrac{1}{x+2}$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{x+h+2}-\dfrac{1}{x+2}}{h} = \lim_{h\to 0} \frac{\dfrac{x+2}{(x+h+2)(x+2)}-\dfrac{x+h+2}{(x+h+2)(x+2)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{x+2-x-h-2}{(x+h+2)(x+2)} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h+2)(x+2)} = \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)} = -\frac{1}{(x+2)^2}
\end{aligned}$$

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