Sample Problems

1. Find the equation of the tangent line drawn to the graph of $-3x^2 - 16xy - 2y^2 + 3y = 178$ at the point $(-3, 5)$.

2. Consider the relation determined by the equation $xy^2 - 5x = 2(y^2 + x^2y - 16)$. Find an equation for all tangent line(s) drawn to the graph of the relation at $x = 3$.

3. If $y = f(x)$ is a function, we define the curvature as

$$C(x) = \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}}$$

Prove that if $f(x) = \sqrt{r^2 - x^2}$ where $r > 0$, then the curvature is constant on the interval $(-r, r)$.

Practice Problems

1. Find the slope of the tangent line drawn to the graph of $x^4 - y^4 = 2x^2y + 23$ to the point $(2, -1)$.

2. Find an equation for the tangent line drawn to the graph of $x^3 + y^3 - 5y^2 = 6x^2 + 13x - 42$ at the point $(-3, 5)$.

3. Find an equation for all tangent lines drawn to the graph of $2x^2 + y^2 = 5y - x$ at $x = -2$.

4. Find an equation of all tangent lines drawn to the curve $x^2 - xy + y^2 = 16$ at $x = 0$.

5. Use implicit differentiation to compute y' in terms of x and y.

a) $2x^2 + 4xy = 10$

b) $x^4 + y^4 = 20y$

c) $x^3 + y^3 = 2xy$

d) $x^3 + y^3 = x^2 + y^2$

e) $\ln x - 2 + y^2 = y^5$

f) $x^2 + y^2 = \dfrac{1}{y}$

g) $\ln x + xy = -2y^3$

h) $x^4y - xy^4 = y$

i) $x^3 + y^3 = (x - y)^5$

j) $y^3 + y = \sqrt{x^2 - y^2}$

k) $2^{x+y} = xy^3$

l) $y + xy = \sqrt{xy - 2}$

m) $\ln y = \sqrt{xy - 1}$

n) $e^{x^2y} = x + y^2$

Sample Problems - Answers

1.) $y = 2x + 11$ 2.) $y = -5x + 32$ and $y = -x + 2$ 3.) see solutions

Practice Problems - Answers

1. 10
2. $-2(x+3) = y - 5$
3. $y = -7x - 12$ and $y = 7x + 17$
4. $y = \frac{1}{2}x + 4$ and $y = \frac{1}{2}x - 4$
5. a) $y' = -\frac{x+y}{x}$ b) $y' = -\frac{x^3}{y^3 - 5}$ c) $y' = \frac{3x^2 - 2y}{2x - 3y^2}$ d) $y' = \frac{-3x^2 + 2x}{3y^2 - 2y}$

   e) $y' = -\frac{1}{x(2y - 5y^4)}$ f) $y' = -\frac{2xy^2}{2y^3 + 1}$ g) $y' = -\frac{1 + xy}{x^2 + 6xy^2}$ h) $y' = \frac{y^4 - 4x^3y}{x^4 - 4xy^3 - 1}$

   i) $y' = \frac{-3x^2 + 5(x-y)^4}{3y^2 + 5(x-y)^4}$ j) $y' = \frac{x}{y + (y + y^3)(3y^2 + 1)}$ k) $y' = \frac{y^3 - (\ln 2)\, 2^{x+y}}{-3xy^2 + (\ln 2)\, 2^{x+y}}$

   l) $y' = \frac{y - 2y\sqrt{xy - 2}}{2\sqrt{xy - 2} - x + 2x\sqrt{xy - 2}}$ m) $y' = -\frac{y^2}{xy - 2\sqrt{xy - 1}}$ n) $y' = \frac{-2xye^{x^2y} + 1}{-2y + x^2e^{x^2y}}$

Sample Problems - Solutions

1. Find the equation of the tangent line drawn to the graph of $-3x^2 - 16xy - 2y^2 + 3y = 178$ at the point $(-3, 5)$.
Solution: We start with implicit differentiation. We first differentiate both sides: Then we solve for y'.

$$\begin{aligned} -3x^2 - 16xy - 2y^2 + 3y &= 178 \\ -6x - 16y - 16xy' - 4yy' + 3y' &= 0 \\ -16xy' - 4yy' + 3y' &= 6x + 16y \\ y'(-16x - 4y + 3) &= 6x + 16y \\ y' &= \frac{6x + 16y}{-16x - 4y + 3} \qquad \text{compute } y' \text{ when } x = -3 \text{ and } y = 5 \\ y' &= \frac{6(-3) + 16(5)}{-16(-3) - 4(5) + 3} = 2 \end{aligned}$$

The line must pass through $(-3, 5)$ and have slope 2.

$$\begin{aligned} y - 5 &= 2(x + 3) \\ y &= 2x + 6 + 5 = 2x + 11 \end{aligned}$$

Thus the answer is $y = 2x + 11$.

2. Consider the relation determined by the equation $xy^2 - 5x = 2(y^2 + x^2y - 16)$. Find an equation for all tangent line(s) drawn to the graph of the relation at $x = 3$.
Solution: We substitute $x = 3$ into the equation and solve for y.

$$\begin{aligned} 3y^2 - 15 &= 2\left(y^2 + 9y - 16\right) \\ 3y^2 - 15 &= 2y^2 + 18y - 32 \\ y^2 - 18y + 17 &= 0 \\ (y - 17)(y - 1) &= 0 \qquad \Longrightarrow y_1 = 17 \qquad y_2 = 1 \end{aligned}$$

Thus there are two points with tangent lines: $(3, 17)$ and $(3, 1)$.
For the slope of each tangent lines, we differentiate both sides and solve for y'.

$$\begin{aligned} xy^2 - 5x &= 2\left(y^2 + x^2y - 16\right) \\ y^2 + x(2yy') - 5 &= 2\left(2yy' + 2xy + x^2y'\right) \\ y^2 + 2xyy' - 5 &= 4yy' + 4xy + 2x^2y' \\ y^2 - 4xy - 5 &= 4yy' + 2x^2y' - 2xyy' \\ y^2 - 4xy - 5 &= y'\left(4y + 2x^2 - 2xy\right) \\ \frac{y^2 - 4xy - 5}{4y + 2x^2 - 2xy} &= y' \end{aligned}$$

The slope of the tangent line drawn to $(3, 17)$

$$m_1 = \frac{y^2 - 4xy - 5}{4y + 2x^2 - 2xy} = \frac{17^2 - 4(3)(17) - 5}{4(17) + 2(3)^2 - 2(3)(17)} = \frac{80}{-16} = -5$$

We can easily find the equation of the line with slope -5, passing through $(3, 17)$; it is $y = -5x + 32$. The other tangent line, passing through $(3, 1)$ has slope

$$m_2 = \frac{y^2 - 4xy - 5}{4y + 2x^2 - 2xy} = \frac{1^2 - 4(3)(1) - 5}{4(1) + 2(3)^2 - 2(3)(1)} = \frac{-16}{16} = -1$$

Thus the slope is -1 and the equation of this line is $y = -x + 2$.

3. If $y = f(x)$ is a function, we define the curvature as

$$C(x) = \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}}$$

Prove that if $f(x) = \sqrt{r^2 - x^2}$ where $r > 0$, then the curvature is constant on the interval $(-r, r)$.
Proof: Let us write y for $f(x)$. We can see that on $(-r, r)$ y is always positive and that $x^2 + y^2 = r^2$

$$\begin{aligned} x^2 + y^2 &= r^2 && \text{differentiate both sides} \\ 2x + 2yy' &= 0 \\ x + yy' &= 0 && \Longrightarrow \quad y' = -\frac{x}{y} \end{aligned}$$

For the second derivative, y'' we differentiate both sides of the statement $x + yy' = 0$

$$\begin{aligned} x + yy' &= 0 \\ 1 + y'y' + yy'' &= 0 \\ 1 + (y')^2 + yy'' &= 0 \\ y'' &= \frac{-1 - (y')^2}{y} = \frac{-1 - \left(-\frac{x}{y}\right)^2}{y} = \frac{-1 - \frac{x^2}{y^2}}{y} = \frac{\frac{-y^2 - x^2}{y^2}}{y} = \frac{-x^2 - y^2}{y^3} = \frac{-r^2}{y^3} \end{aligned}$$

Notice that since y is always positive, $y'' = \dfrac{-r^2}{y^3}$ is always negative. Thus $|y''| = -y''$.

$$\begin{aligned} C(x) &= \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}} = \frac{-y''}{\left(1 + (y')^2\right)^{3/2}} = \frac{\frac{r^2}{y^3}}{\left(1 + \left(-\frac{x}{y}\right)^2\right)^{3/2}} = \frac{\frac{r^2}{y^3}}{\left(1 + \frac{x^2}{y^2}\right)^{3/2}} \\ &= \frac{\frac{r^2}{y^3}}{\left(\frac{y^2 + x^2}{y^2}\right)^{3/2}} = \frac{\frac{r^2}{y^3}}{\left(\frac{r^2}{y^2}\right)^{3/2}} = \frac{\frac{r^2}{y^3}}{\frac{r^3}{y^3}} = \frac{1}{r} \end{aligned}$$

