Theorem: $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ for all $x > 0$

Proof: Let $x > 0$ be fixed.

$$\begin{aligned}
\frac{d}{dx}(\ln x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\dfrac{x+h}{x}}{h} = \lim_{h\to 0}\frac{1}{h}\cdot\ln\frac{x+h}{x} \\
&= \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{1/h} = \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h} = \lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{1}{h}} = \lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}\cdot\frac{1}{x}} \\
&= \lim_{h\to 0}\ln\left[\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} = \lim_{h\to 0}\left[\frac{1}{x}\cdot\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right]
\end{aligned}$$

Since $\dfrac{1}{x}$ is a constant, we can write

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0}\left(\frac{1}{x}\cdot\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right) = \frac{1}{x}\cdot\lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}$$

Define $m = \dfrac{x}{h}$. Since $x > 0$ is fixed, approaches infinity as h approaches 0 from the right and negative infinity as h approaches 0 from the left. Then

$$\frac{d}{dx}(\ln x) = \frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}} = \frac{1}{x}\lim_{m\to\infty}\ln\left(1+\frac{1}{m}\right)^{m}$$

Since $f(x) = \ln x$ is continuous at e, we also have that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}\lim_{m\to\infty}\ln\left(1+\frac{1}{m}\right)^{m} = \frac{1}{x}\ln\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = \frac{1}{x}\ln e = \frac{1}{x}\cdot 1 = \frac{1}{x}$$

Theorem: $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x\ln a}$ for all $a > 0$, $a \neq 1$

Proof: We will use the change-base theorem.

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{d}{dx}\left(\frac{1}{\ln a}\cdot\ln x\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$