Theorem 1: If f is differentiable at a, then it is continuous there.

Proof: Suppose that f is differentiable at a number a. Then $f'(a)$ exists which means that $f(a)$ exists and the limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ also exists and is finite. Let us start with the true statement that $0 = 0\cdot f'(a)$.

$$\begin{aligned}
0 &= 0\cdot f'(a) \\
0 &= \lim_{h\to 0} h\cdot \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{by the product rule of limits} \\
0 &= \lim_{h\to 0}\left(h\cdot\frac{f(a+h)-f(a)}{h}\right) && \text{cancel out } h \\
0 &= \lim_{h\to 0}\left(f(a+h)-f(a)\right) && \text{by the difference rule of limits} \\
0 &= \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) \\
\lim_{h\to 0} f(a) &= \lim_{h\to 0} f(a+h) && \text{by the constant rule of limits} \\
f(a) &= \lim_{h\to 0} f(a+h)
\end{aligned}$$

and $f(a) = \lim_{h\to 0} f(a+h)$ means that f is continuous at a.

Theorem 2: (Product rule of derivatives) If f and g are differentiable at x, then so is fg and the derivative is $(fg)' = f'g + fg'$.

Proof: Suppose that f and g are differentiable at x.

$$\begin{aligned}
(fg)' &= \lim_{h\to 0}\frac{fg(x+h)-fg(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} && \text{smuggle in } f(x+h)g(x) \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)\left[g(x+h)-g(x)\right]+g(x)\left[f(x+h)-f(x)\right]}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} && \text{by the sum rule of limits} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h} && \text{by the product rule of limits} \\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{by the constant rule of limits} \\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}
\end{aligned}$$

We now realize the following:

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x) \quad \text{and} \quad \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Furthermore, f is continuous because it is differentiable, and so $\lim_{h\to 0} f(x+h) = f(x)$. Thus we now have that

$$\begin{aligned}
(fg)' &= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= f(x) \cdot g'(x) + g(x)\cdot f'(x)
\end{aligned}$$

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