Theorem: Suppose that f and g are differentiable functions with $g \neq 0$. Then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{\frac{f}{g}(x+h) - \frac{f}{g}(x)}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

We smuggle in $f(x)g(x)$

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x) - f(x)g(x)}{hg(x)g(x+h)} + \frac{f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}\right)$$

We factor out $g(x)$ in the first term and $f(x)$ in the second.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\left(\frac{g(x)[f(x+h) - f(x)]}{hg(x)g(x+h)} + \frac{f(x)[g(x) - g(x+h)]}{hg(x)g(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{hg(x+h)} + \lim_{h\to 0}\frac{f(x)[g(x) - g(x+h)]}{hg(x)g(x+h)}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)} + f(x)\lim_{h\to 0}\frac{-[g(x+h) - g(x)]}{hg(x)g(x+h)}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)} + f(x)\lim_{h\to 0}\left(-\frac{g(x+h) - g(x)}{h}\right)\lim_{h\to 0}\frac{1}{g(x)g(x+h)}$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot(-g'(x))\cdot\frac{1}{(g(x))^2}$$

This last conclusion is correct because of the following facts:

$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x)$ by the definition of the derivative

$\lim_{h\to 0}\frac{1}{g(x+h)} = \frac{1}{\lim_{h\to 0} g(x+h)} = \frac{1}{g(x)}$ because g is continuous at x

$\lim_{h\to 0}\left(-\frac{g(x+h) - g(x)}{h}\right) = -\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = -g'(x)$ by properties of limits and the definition of the derivative.

$\lim_{h\to 0}\frac{1}{g(x)g(x+h)} = \frac{1}{\lim_{h\to 0} g(x)g(x+h)} = \frac{1}{g(x)\lim_{h\to 0} g(x+h)} = \frac{1}{(g(x))^2}$ because g is continuous.

Thus we have that

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} + \frac{f(x)(-g'(x))}{(g(x))^2} = \frac{f'(x)g(x)}{(g(x))^2} + \frac{f(x)(-g'(x))}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

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