Definition: We define the natural logarithm function as $\ln x = \int_1^x \frac{1}{t}\, dt$ **with domain** $(0, \infty)$

1. On its domain $(0, \infty)$, $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\ln x$ is continuous, strictly increasing, and one-to-one.

   proof: On its domain $(0, \infty)$, $\frac{d}{dx} \ln x = \frac{1}{x}$ by the fundamental theorem

   $\ln x$ is continuous because it is differentiable. It is strictly increasing because its derivative, $\frac{1}{x}$ is always positive. Finally, $\ln x$ is one-to-one because it is strictly increasing.

2. $\ln 1 = 0$, $\ln x$ is positive for $x > 1$ and $\ln x$ is negative for $0 < x < 1$

   proof: $\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0$

   $\frac{1}{x}$ is positive on $(0, \infty)$ and so if $x > 1$, then $\int_1^x \frac{1}{t}\, dt$ is positive

   Suppose that $0 < x < 1$. Then $\int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt$ is negative

3. a) $\ln xy = \ln x + \ln y$ for all positive x, y

   proof: Fix y and define the function $f(x) = \ln xy - \ln x$

$$f'(x) = (\ln xy - \ln x)' = \frac{1}{xy}(y) - \frac{1}{x} = 0$$

   Since its derivative is zero, $f(x)$ is constant. $f(1) = \ln y - \ln 1 \stackrel{2}{=} \ln y - 0$
   So for all x, $f(x) = \ln y$

$$\begin{aligned} \ln y &= \ln xy - \ln x \\ \ln x + \ln y &= \ln xy \end{aligned}$$

3. b) $\ln \frac{1}{x} = -\ln x$ for all positive x

   Now that we have proven $\ln xy = \ln x + \ln y$, let $y = \frac{1}{x}$. Then

$$\begin{aligned} 0 \stackrel{2}{=} \ln 1 &= \ln x\left(\frac{1}{x}\right) \stackrel{3a}{=} \ln x + \ln \frac{1}{x} \\ 0 &= \ln x + \ln \frac{1}{x} \\ -\ln x &= \ln \frac{1}{x} \end{aligned}$$

4. $\ln \frac{x}{y} = \ln x - \ln y$

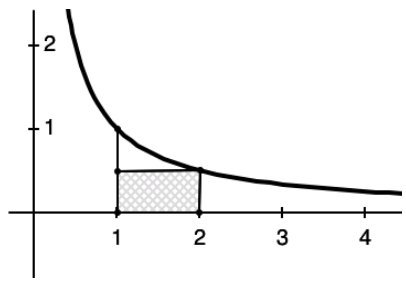
   proof:

$$\ln \frac{x}{y} = \ln\left(x\left(\frac{1}{y}\right)\right) \stackrel{3a}{=} \ln x + \ln \frac{1}{y} \stackrel{3b}{=} \ln x + (-\ln y) = \ln x - \ln y$$

5. $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$. The range of $\ln x$ is $(-\infty, \infty)$.

Claim 1: $\ln 2 > \frac{1}{2}$

proof: Consider the picture below. As it shows, $\ln 2$ is the area under the graph of $\frac{1}{x}$ on $[1, 2]$. The area of the rectangle inside this region is $\frac{1}{2}$.



Claim 2: For any positive number N, there exists x with $\ln x > N$.
proof: Let $x = 2^{2N}$.

$$\ln x = \ln 2^{2N} = \ln (2 \cdot 2 \cdot 2 ... \cdot 2) = \ln 2 + \ln 2 + \ln 2 + ... + \ln 2 = (2N) \ln 2 > (2N) \left(\frac{1}{2}\right) = N$$

Thus $\lim_{x\to\infty} \ln x = \infty$.

For large negative values of the natrual logarithm function, let $x = 2^{2N}$ as before. Then we already have shown that $\ln x > N$ and also,

$$\ln \frac{1}{x} = -\ln x < -N$$

Thus $\lim_{x\to 0^+} \ln x = -\infty$.

For the range: We proved that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$. Since $\ln x$ is continuous, by the intermediate value theorem, the function takes every real number as a function value.

Definition: We define the natural exponent $f(x) = \exp x$ as the inverse function of $\ln x$. Later on, we will denote it as e^x.

6. $\exp x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$

proof: Recall that $\ln x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$. $\exp e$, the inverse will have those reversed.

7. $\exp(\ln x) = x$ for all positive x and $\ln(\exp y) = y$ for all positive y.

(Same as $e^{\ln x} = x$ for all positive x and $\ln e^y = y$ for all positive y)

proof: follows from the definition of inverse functions.

8. $\exp(x + y) = (\exp x)(\exp y)$ for all x, y

(Same as $e^{x+y} = e^x e^y$ for all x, y)

proof:

$$\exp x \exp y \overset{7}{=} \exp(\ln(\exp x \exp y)) \overset{3a}{=} \exp(\ln \exp x + \ln \exp y) \overset{7}{=} \exp(x + y)$$

9. $\exp(x-y) = \dfrac{\exp x}{\exp y}$ for all x, y

(Same as $e^{x-y} = \dfrac{e^x}{e^y}$ for all x, y)

proof:

$$\frac{\exp x}{\exp y} \overset{7}{=} \exp\left(\ln\left(\frac{\exp x}{\exp y}\right)\right) \overset{4}{=} \exp\left(\ln(\exp x) - \ln(\exp y)\right) \overset{7}{=} \exp(x-y)$$

10. On its domain $(-\infty, \infty)$, $\exp x$ is strictly increasing, one-to-one, and $\dfrac{d}{dx}\exp x = \exp x$.

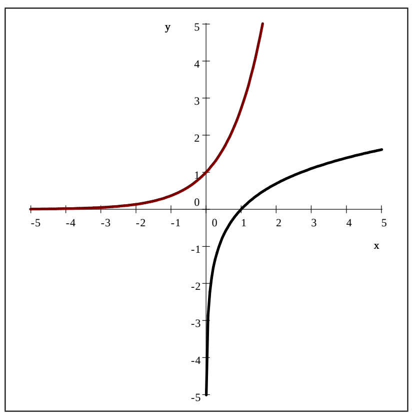
proof: For all positive x, $\ln(\exp x) \overset{7}{=} x$

$$\begin{aligned} \ln(\exp x) &= x && \text{differentiate both sides} \\ \frac{1}{\exp x}\cdot\frac{d}{dx}\exp x &= 1 && \text{multiply by } \exp x \\ \frac{d}{dx}\exp x &= \exp x \end{aligned}$$

Since the derivative of $\exp x$ is always positive, $\exp x$ is always strictly increasing and therefore also one-to-one.

11. $\lim\limits_{x\to\infty} \exp x = \infty$ and $\lim\limits_{x\to-\infty} \exp x = 0$

Recall that $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$ and that $\ln x$ and $\exp x$ are inverse functions.



Definition: Define e to be the number $e = \exp 1$.

12. $\ln e = 1$

proof: $\ln e \overset{\text{def}}{=} \ln(\exp 1) \overset{7}{=} 1$

Definition: For positive b, let $\exp_b x = \exp(x \ln b)$. Later we will use the notation b^x for $\exp_b x$.

13. $\exp_e x = \exp x$ for all x

proof: $\exp_e x \overset{\text{def}}{=} \exp(x \ln e) \overset{12}{=} \exp(x \cdot 1) = \exp x$

14. a) $\exp_b 0 = 1$ (same as $b^0 = 1$)

proof: $\exp_b 0 \overset{\text{def}}{=} \exp(0 \ln b) = \exp 0 \overset{2}{=} \exp(\ln 1) \overset{7}{=} 1$

14. b) $\exp_b 1 = b$ (same as $b^1 = b$)

proof: $\exp_b 1 \overset{\text{def}}{=} \exp(1 \ln b) = \exp(\ln b) \overset{\text{def}}{=} b$

14. c) $\exp_b(-1) = \dfrac{1}{b}$ (same as $b^{-1} = \dfrac{1}{b}$) (Recall that $b > 0$)

proof: $\exp_b(-1) \overset{\text{def}}{=} \exp(-1 \ln b) \overset{3b}{=} \exp\left(\ln\left(\dfrac{1}{b}\right)\right) \overset{7}{=} \dfrac{1}{b}$

14. d) $\exp_1 x = 1$ (same as $1^x = 1$)

proof: $\exp_1 x \overset{\text{def}}{=} \exp(x \ln 1) \overset{2}{=} \exp(x \cdot 0) = \exp 0 \overset{14a}{=} 1$

15. $\ln(\exp_b x) = x \ln b$ for all x and all positive b

(same as $\ln b^x = x \ln b$ for all x and all positive b)

proof: $\ln(\exp_b x) \overset{\text{def}}{=} \ln(\exp(x \ln b)) \overset{7}{=} x \ln b$

16. a) $\exp_b(x+y) = (\exp_b x)(\exp_b y)$ for all x, y and all positive b

(same as $b^{x+y} = b^x b^y$ for all x and all positive b)

proof: $\exp_b(x+y) \overset{\text{def}}{=} \exp((x+y)\ln b) = \exp(x \ln b + y \ln b) \overset{8}{=} (\exp(x \ln b))(\exp(y \ln b)) \overset{\text{def}}{=} (\exp_b x)(\exp_b y)$

16. b) $\exp_{\exp_b x} y = \exp_b(xy)$ for all x, y and all positive b

(same as $(b^x)^y = b^{xy}$ for all x, y and all positive b)

proof: $\exp_{\exp_b x} y \overset{\text{def}}{=} \exp(y \ln(\exp_b x)) \overset{\text{def}}{=} \exp(y \ln(\exp(x \ln b))) \overset{7}{=} \exp(yx \ln b) \overset{\text{def}}{=} \exp_b(xy)$

17. a) $\exp_b(-x) = \dfrac{1}{\exp_b x}$ for all x and for all positive b

(same as $b^{-x} = \dfrac{1}{b^x}$ for all x and all positive b)

proof:

$$
\begin{aligned}
1 \overset{14a}{=} \exp_b 0 &= \exp_b(x + (-x)) \overset{16a}{=} (\exp_b x)(\exp_b(-x)) \\
1 &= \exp_b x \cdot (\exp_b(-x)) \quad \Longrightarrow \quad \exp_b(-x) = \frac{1}{\exp_b x}
\end{aligned}
$$

17. b) $\exp_{bc} x = (\exp_b x)(\exp_c x)$ for all x and all positive b and c

(same as $(bc)^x = b^x c^x$ for all x and all positive b and c)

proof:

$\exp_{bc} x \overset{\text{def}}{=} \exp(x \ln bc) \overset{3a}{=} \exp(x(\ln b + \ln c)) = \exp(x \ln b + x \ln c) \overset{8}{=} (\exp(x \ln b))(\exp(x \ln c)) \overset{\text{def}}{=} (\exp_b x)(\exp_c x)$

18. a) $\exp_b(x-y) = \dfrac{\exp_b x}{\exp_b y}$ for all x, y and all positive b

(Same as $b^{x-y} = \dfrac{b^x}{b^y}$ for all x, y and all positive b)

proof: $\exp_b(x-y) = \exp_b(x+(-y)) \overset{16a}{=} (\exp_b x)(\exp_b(-y)) \overset{17a}{=} \exp_b x \dfrac{1}{\exp_b y} = \dfrac{\exp_b x}{\exp_b y}$

19. a) $b^n = \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$ for all positive b and natural number n

proof: Write $n = \underbrace{1+1+\ldots+1}_{n \text{ times}}$

$$b^n = \exp_b n = \exp_b \underbrace{(1+1+\ldots+1)}_{n \text{ times}} \overset{16a}{=} \underbrace{(\exp_b 1)(\exp_b 1)\ldots(\exp_b 1)}_{n \text{ times}} \overset{14b}{=} \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

Please note that a rigorous mathematical proof of this statement requires a proving technique called induction.

Let us use the usual notation: Let $b^x = \exp_b x$ and $e^x = \exp x$.

19. b) $b^{1/m} = \sqrt[m]{b}$ for all positive b and positive integers m

proof: $\underbrace{\dfrac{1}{m} + \dfrac{1}{m} + \ldots + \dfrac{1}{m}}_{m \text{ times}} = 1$

$$\underbrace{\left(\exp_b \frac{1}{m}\right)\left(\exp_b \frac{1}{m}\right)\ldots\left(\exp_b \frac{1}{m}\right)}_{m \text{ times}} \overset{14b}{=} \exp_b \underbrace{\left(\frac{1}{m} + \frac{1}{m} + \ldots + \frac{1}{m}\right)}_{m \text{ times}} = \exp_b 1 = b$$

By definition, $b^{1/m} = \exp_b \dfrac{1}{m}$ is a positive number. By the computation above, $\exp_b \dfrac{1}{m}$ is a number such that when it is raised to the mth power, the result is b. Thus, it is $\sqrt[m]{b}$.

19. c) $b^{n/m} = \sqrt[m]{b^n}$

proof: We write $\dfrac{n}{m} = n\left(\dfrac{1}{m}\right)$

$$b^{n/m} = \exp_b\left(\frac{n}{m}\right) = \exp_b\left(n\left(\frac{1}{m}\right)\right) \overset{16b}{=} \exp_{\exp_b n}\left(\frac{1}{m}\right) \overset{19a}{=} \exp_{b^n}\left(\frac{1}{m}\right) \overset{19b}{=} \sqrt[m]{b^n}$$

20. Let $b > 0$ be fixed. Then $\dfrac{d}{dx} b^x = b^x \ln b$ on its domain $(-\infty, \infty)$.

proof: By the Chain rule,

$$\frac{d}{dx} b^x \overset{\text{def}}{=} \frac{d}{dx} \exp(x \ln b) \overset{10}{=} \exp(x \ln b) \frac{d}{dx}(x \ln b) = \exp(x \ln b) \ \ln b = b^x \ \ln b$$

21. Let t be any real number, fixed. Then $\dfrac{d}{dx} x^t = t x^{t-1}$ on its domain $(0, \infty)$.

proof:

$$\frac{d}{dx} x^t = \frac{d}{dx} \exp(t \ln x) \overset{10}{=} \exp(t \ln x) \frac{d}{dx}(t \ln x) = x^t \frac{t}{x} = t \frac{x^t}{x} = t x^{t-1}$$

22. $b^{\ln x/\ln b} = x$ for all positive x and all positive b with $b \neq 1$. That is, $\exp_b\left(\frac{\ln x}{\ln b}\right) = x$

proof: $\exp_b\left(\frac{\ln x}{\ln b}\right) \stackrel{\text{def}}{=} \exp\left(\ln b \frac{\ln x}{\ln b}\right) = \exp(\ln x) \stackrel{7}{=} x$

23. For all positive b with $b \neq 1$, b^x is one-to-one and has domain $(-\infty, \infty)$ and range $(0, \infty)$.

proof: $b^x = \exp(x \ln b)$ is positive for all x since $\exp x$ has range $(0, \infty)$.

To show that the range is all of $(0, \infty)$, let $y > 0$ be given. Let $x = \frac{\ln y}{\ln b}$. Then $b^x = b^{\ln y/\ln b} \stackrel{22}{=} y$. the domain of $b^x = \exp(x \ln b)$ is $(-\infty, \infty)$ since that is the domain of $\exp x$ (see in 6).

Recall from 20 that $\frac{d}{dx} b^x = b^x \ln b$. Since b^x is positive for all x, the expression $b^x \ln b$ is positive for $b > 1$ and negative for $0 < b < 1$. Thus b^x is strictly increasing if $b > 1$ (because its derivative is positive) and strictly decreasing if $0 < b < 1$ (becasue its derivative is negative). In both cases, the function b^x is one-to-one.

Definition: For all positive b with $b \neq 1$, we define $\log_b x$ to be the inverse function of $\exp_b x$.

24. For positive b with $b \neq 1$, $\log_b x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.

proof: This follows from the definition of inverse functions and 23.

25. For positive b with $b \neq 1$, $b^{\log_b x} = x$ for all positive x and $\log_b b^y = y$ for all y.

proof: This follows from the definition of inverse functions.

26. $\log_b x = \frac{\ln x}{\ln b}$ for all b and x with positive values and with $b \neq 1$.

proof: Recall from 22 that $b^{\ln x/\ln b} = x$ and take the base-b logarithm of both sides.

$$\begin{aligned} x &= b^{\ln x/\ln b} \\ \log_b x &= \log_b\left(b^{\ln x/\ln b}\right) \stackrel{25}{=} \frac{\ln x}{\ln b} \end{aligned}$$

27. a) $\log_e x = \ln x$ for all positive x

proof: We start with 26:

$$\log_e x \stackrel{26}{=} \frac{\ln x}{\ln e} \stackrel{12}{=} \frac{\ln x}{1} = \ln x$$

27. b) $\log_b x = \frac{\log_c x}{\log_c b}$ for all b, c, and x positive with $b \neq 1$ and $c \neq 1$.

proof: We will use 26.

$$\frac{\log_c x}{\log_c b} \stackrel{26}{=} \frac{\frac{\ln x}{\ln c}}{\frac{\ln b}{\ln c}} = \frac{\ln x}{\ln c}\frac{\ln c}{\ln b} = \frac{\ln x}{\ln b} \stackrel{26}{=} \log_b x$$

28. a) $\log_b 1 = 0$ for all positive b with $b \neq 1$.

proof: $\log_b 1 \overset{26}{=} \dfrac{\ln 1}{\ln b} \overset{2}{=} \dfrac{0}{\ln b} = 0$

28. b) $\log_b b = 1$ for all positive b with $b \neq 1$.

proof: $\log_b b \overset{26}{=} \dfrac{\ln b}{\ln b} = 1$

29. a) $\log_b xy = \log_b x + \log_b y$ for all positive b, x, y with $b \neq 1$.

proof: $\log_b x + \log_b y \overset{26}{=} \dfrac{\ln x}{\ln b} + \dfrac{\ln y}{\ln b} = \dfrac{\ln x + \ln y}{\ln b} \overset{3}{=} \dfrac{\ln xy}{\ln b} \overset{26}{=} \log_b xy$

29. b) $\log_b x^t = t \log_b x$ for all positive b, x with $b \neq 1$ and any t.

proof: $\log_b x^t \overset{26}{=} \dfrac{\ln x^t}{\ln b} = \dfrac{\ln(\exp_t x)}{\ln b} \overset{15}{=} \dfrac{t \ln x}{\ln b} = t\dfrac{\ln x}{\ln b} \overset{26}{=} t \log_b x$

30. For all positive b with $b \neq 1$, $\dfrac{d}{dx} \log_b x = \dfrac{1}{x \ln b}$ on its domain $(0, \infty)$.

proof: Recall from 26 that $\log_b x = \dfrac{\ln x}{\ln b}$.

$$\frac{d}{dx} \log_b x \overset{26}{=} \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b} \frac{d}{dx}(\ln x) \overset{1}{=} \frac{1}{\ln b}\frac{1}{x} = \frac{1}{x \ln b}$$

Based on a writeup by Michael Maltenfort.