Definition: We define the hyperbolic functions as

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Theorem: $\boxed{\sinh 0 = 0 \text{ and } \cosh 0 = 1}$

Proof:

$$\sinh 0 = \frac{1}{2}\left(e^0 - e^{-0}\right) = \frac{1}{2}(1-1) = 0 \quad \text{and} \quad \cosh 0 = \frac{1}{2}\left(e^0 + e^{-0}\right) = \frac{1}{2}(1+1) = 1$$

Theorem: $\boxed{\sinh x \text{ and } \tanh x \text{ are odd, and } \cosh x \text{ is even.}}$

Proof: Let x be any real number.

$$\begin{aligned}
\sinh(-x) &= \frac{1}{2}\left(e^{-x} - e^{-(-x)}\right) = \frac{1}{2}\left(e^{-x} - e^{x}\right) = -\frac{1}{2}\left(e^{x} - e^{-x}\right) = -\sinh x \\
\cosh(-x) &= \frac{1}{2}\left(e^{-x} + e^{-(-x)}\right) = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cosh x
\end{aligned}$$

$$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$$

Limits

Theorem: $\boxed{\lim_{x\to\infty} \sinh x = \infty \quad \text{and} \quad \lim_{x\to-\infty} \sinh x = -\infty}$

Proof: $\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{1}{2}\left(\underset{\infty}{\underset{\downarrow}{e^x}} - \underset{0}{\underset{\downarrow}{e^{-x}}}\right) = \infty$ Since $\sinh x$ is odd, $\lim_{x\to-\infty} \sinh x = -\infty$ follows.

Theorem: $\boxed{\lim_{x\to\infty} \cosh x = \lim_{x\to-\infty} \cosh x = \infty}$

Proof: $\lim_{x\to\infty} \cosh x = \lim_{x\to\infty} \frac{1}{2}\left(\underset{\infty}{\underset{\downarrow}{e^x}} + \underset{0}{\underset{\downarrow}{e^{-x}}}\right) = \infty$ Since $\cosh x$ is even, $\lim_{x\to-\infty} \cosh x = \infty$ follows.

Theorem: $\boxed{\lim_{x\to\infty} \tanh x = 1 \quad \text{and} \quad \lim_{x\to-\infty} \tan x = -1}$

Proof:

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x\left(1 - \frac{1}{e^{2x}}\right)}{e^x\left(1 + \frac{1}{e^{2x}}\right)} = \lim_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = 1$$

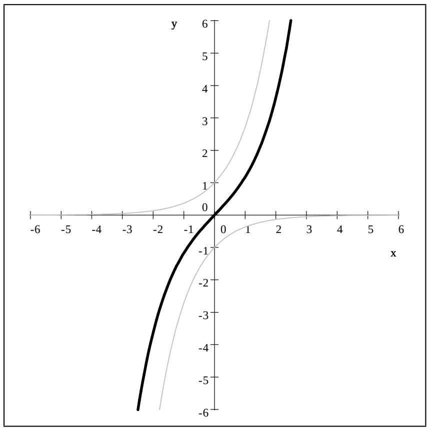
Since $\tanh x$ is odd, $\lim_{x\to-\infty} \tanh x = -1$ follows.

Graphs

$y = \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

domain: $\mathbb{R}$
range: $\mathbb{R}$

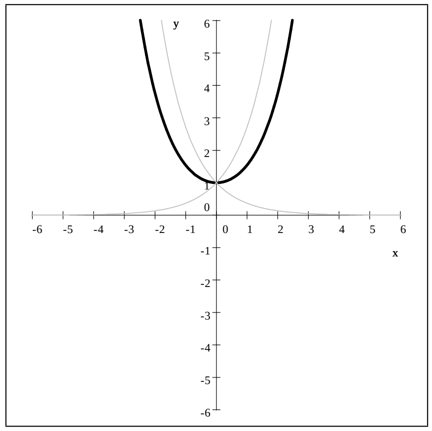
The grey graphs are
$y = e^x$ and $y = -e^{-x}$.



$y = \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

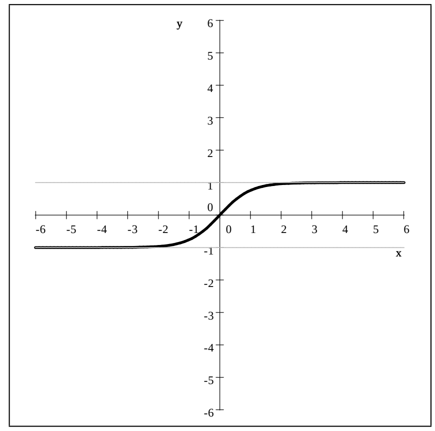
domain: $\mathbb{R}$
range: $[1, \infty)$

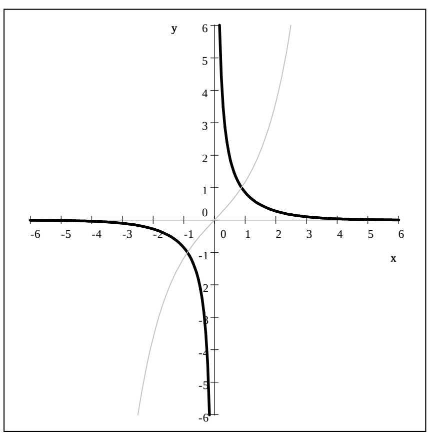
The grey graphs are
$y = e^x$ and $y = e^{-x}$.



$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

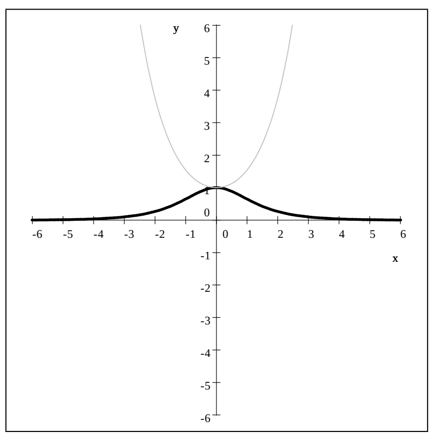
domain: $\mathbb{R}$
range: $(-1, 1)$





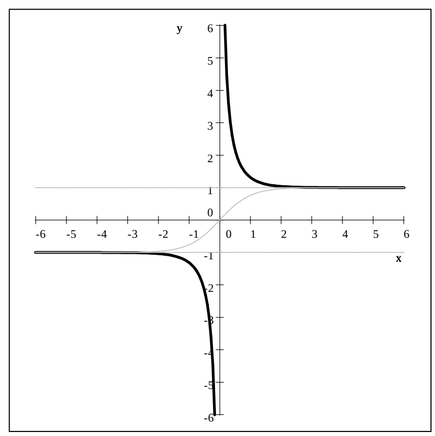
$y = \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

domain: $(-\infty, 0) \cup (0, \infty)$
range: $(-\infty, 0) \cup (0, \infty)$



$y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

domain: $\mathbb{R}$
range: $(0, 1]$



$y = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

domain: $(-\infty, 0) \cup (0, \infty)$
range: $(-\infty, -1) \cup (1, \infty)$

Identities

Theorem: $\boxed{\cosh^2 x - \sinh^2 x = 1}$ for all real numbers x. Also, $\boxed{\tanh^2 x = 1 - \operatorname{sech}^2 x \text{ and } \coth^2 x = 1 + \operatorname{csch}^2 x}$

Proof: Let x be any real number.

$$\cosh^2 x - \sinh^2 x = \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2 - \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) - \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) = \frac{1}{4}(4) = 1$$

We can derive additional identities from $\cosh^2 x - \sinh^2 x = 1$ by dividing by $\cosh^2 x$ or by $\sinh^2 x$:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} &= \frac{1}{\cosh^2 x} \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \\ 1 - \operatorname{sech}^2 x &= \tanh^2 x \end{aligned}$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x}$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Theorem: ("double-angle" formulas) $\boxed{\sinh 2x = 2\sinh x \cosh x \quad \text{and} \quad \cosh 2x = \cosh^2 x + \sinh^2 x}$ for all real numbers x.

Proof: Let x be any real number.

$$\begin{aligned} 2\sinh x \cosh x &= 2 \cdot \frac{1}{2}\left(e^x - e^{-x}\right)\frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \sinh 2x \\ \cosh^2 x + \sinh^2 x &= \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2 + \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) + \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) \\ &= \frac{1}{4}\left(2e^{2x} + 2e^{-2x}\right) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh 2x \end{aligned}$$

The double-angle formula for $\cosh x$ also has three forms. Recall that $\cosh^2 x - \sinh^2 x = 1$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$$

From this, we can solve for $\sinh^2 x$ and $\cosh^2 x$:

$$\boxed{\cosh^2 x = \frac{1}{2}(\cosh 2x + 1) \quad \text{and} \quad \sinh^2 x = \frac{1}{2}(\cosh 2x - 1)}$$

Derivatives

Theorem: $\boxed{\dfrac{d}{dx}\sinh x = \cosh x \quad \text{and} \quad \dfrac{d}{dx}\cosh x = \sinh x}$

Proof:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = \frac{1}{2}\left(e^x - \left(-e^{-x}\right)\right) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x \quad \text{and}$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) = \frac{1}{2}\left(e^x + \left(-e^{-x}\right)\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

Theorem: $\boxed{\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x \quad \text{and} \quad \dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x}$

Proof:

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\left(\frac{d}{dx}\sinh x\right)\cosh x - \sinh x\left(\frac{d}{dx}\cosh x\right)}{\cosh^2 x} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\coth x &= \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\left(\frac{d}{dx}\cosh x\right)\sinh x - \cosh x\left(\frac{d}{dx}\sinh x\right)}{\sinh^2 x} = \frac{\sinh x \cdot \sinh x - \cosh x \cdot \cosh x}{\sinh^2 x} \\ &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-\left(\cosh^2 x - \sinh^2 x\right)}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x\end{aligned}$$

Theorem: $\boxed{\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \quad \text{and} \quad \dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x}$

Proof:

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}(\cosh x)^{-1} = (-1)(\cosh x)^{-2}\left(\frac{d}{dx}\cosh x\right) = -\frac{1}{\cosh^2 x}(\sinh x) = -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}(\sinh x)^{-1} = (-1)(\sinh x)^{-2}\left(\frac{d}{dx}\sinh x\right) = -\frac{1}{\sinh^2 x}(\cosh x) = -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$$

Antiderivatives

Theorem: $\boxed{\int \sinh x \ dx = \cosh x + C \quad \text{and} \quad \int \cosh x \ dx = \sinh x + C}$

Proof: These follow from the differentiation formulas.

Theorem: $\boxed{\int \tanh x \ dx = \ln(\cosh x) + C \quad \text{and} \quad \int \coth x \ dx = \ln|\sinh x| + C}$

Proof: We compute $\int \tanh x \ dx$ first. Let $u = \cosh x$. Then $du = \sinh x \ dx$ and so

$$\int \tanh x \ dx = \int \frac{\sinh x}{\cosh x} \ dx = \int \frac{1}{\cosh x} \ (\sinh x \ dx) = \int \frac{1}{u} \ du = \ln|u| + C = \ln|\cosh x| + C$$

Since $\cosh x$ is always positive, we do not need the absolute value sign.

$$\int \tanh x \ dx = \ln(\cosh x) + C$$

Now for $\int \coth x \ dx$: Let $u = \sinh x$. Then $du = \cosh x \ dx$ and so

$$\int \coth x \ dx = \int \frac{\cosh x}{\sinh x} \ dx = \int \frac{1}{\sinh x} \ (\cosh x \ dx) = \int \frac{1}{u} \ du = \ln|u| + C = \ln|\sinh x| + C$$

Inverse Hyperbolic Functions

We can easily define $\sinh^{-1} x$ because $\sinh x$ is one-to-one. For $\cosh^{-1} x$, we will need to restrict the domain of $\cosh x$ to $[0, \infty)$. $\tanh x$ is also one-to-one, so $\tanh^{-1} x$ is easily defined.

$y = \sinh^{-1} x$

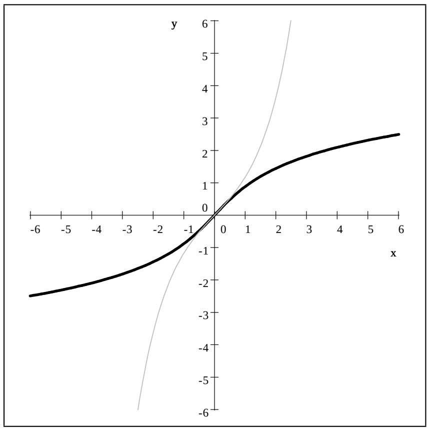
domain: $\mathbb{R}$
range: $\mathbb{R}$

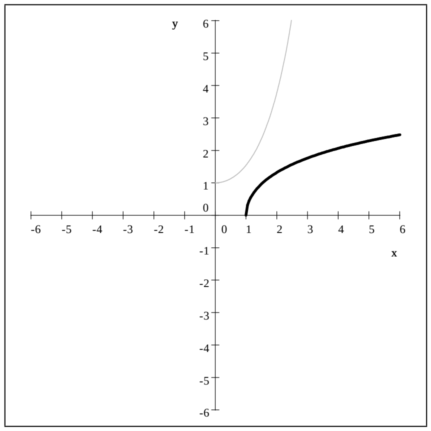
$y = \cosh^{-1} x$

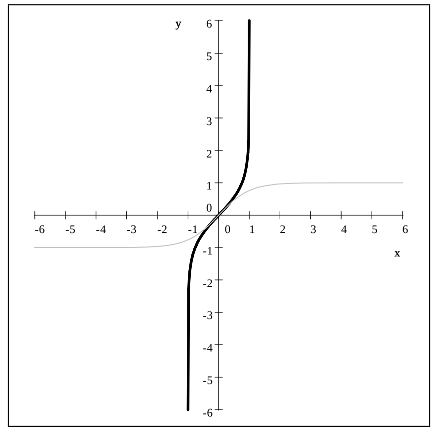
domain: $[1, \infty)$
range: $[0, \infty)$

$y = \tanh^{-1} x$

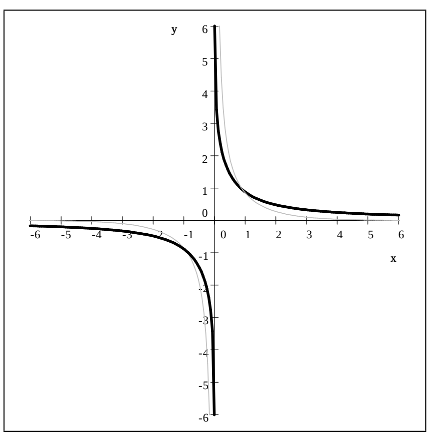
domain: $(-1, 1)$
range: $\mathbb{R}$

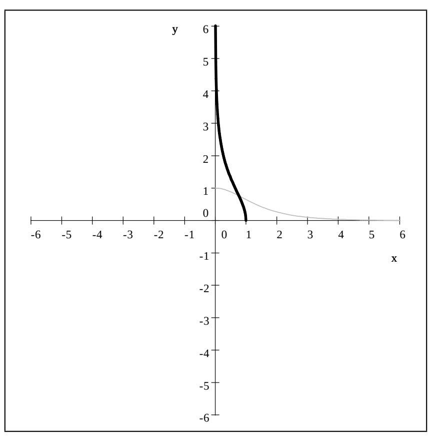


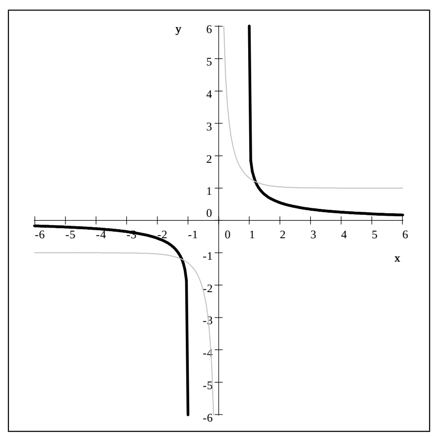




Similarly, the domain of $\operatorname{sech} x$ needs to be restricted to $[0, \infty)$. Since $\operatorname{csch} x$ and $\coth x$ are one-to-one, the inverse functions automatically exist.







$y = \operatorname{csch}^{-1} x$

domain: $(-\infty, 0) \cup (0, \infty)$
range: $(-\infty, 0) \cup (0, \infty)$

$y = \operatorname{sech}^{-1} x$

domain: $(0, 1]$
range: $[0, \infty)$

$y = \coth^{-1} x$

domain: $(-\infty, -1) \cup (1, \infty)$
range: $(-\infty, 0) \cup (0, \infty)$

Identities

Theorem: $\boxed{\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}}$

Proof:

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

We consider $x = \operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)$ and apply sech^{-1} to both sides.

$$\begin{aligned} x &= \operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) \\ \operatorname{sech}^{-1} x &= \operatorname{sech}^{-1}\left(\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)\right) \\ \operatorname{sech}^{-1} x &= \cosh^{-1}\left(\frac{1}{x}\right) \end{aligned}$$

Theorem: $\boxed{\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}}$

Proof:

$$\operatorname{csch}\left(\sinh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sinh\left(\sinh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

Now that we have $x = \operatorname{csch}\left(\sinh^{-1}\left(\frac{1}{x}\right)\right)$, we apply csch^{-1} to both sides.

$$\begin{aligned} x &= \operatorname{csch}\left(\sinh^{-1}\left(\frac{1}{x}\right)\right) \\ \operatorname{csch}^{-1} x &= \operatorname{csch}^{-1}\left(\operatorname{csch}\left(\sinh^{-1}\left(\frac{1}{x}\right)\right)\right) \\ \operatorname{csch}^{-1} x &= \sinh^{-1}\left(\frac{1}{x}\right) \end{aligned}$$

Theorem: $\boxed{\coth^{-1} x = \tanh^{-1} \frac{1}{x}}$

Proof:

$$\coth\left(\tanh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\tanh\left(\tanh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

We consider $x = \coth\left(\tanh^{-1}\left(\frac{1}{x}\right)\right)$ and apply $\coth^{-1}$ to both sides.

$$\begin{aligned} x &= \coth\left(\tanh^{-1}\left(\frac{1}{x}\right)\right) \\ \coth^{-1} x &= \coth^{-1}\left(\coth\left(\tanh^{-1}\left(\frac{1}{x}\right)\right)\right) \\ \coth^{-1} x &= \tanh^{-1}\left(\frac{1}{x}\right) \end{aligned}$$

Derivatives of inverse functions

Theorem: $\boxed{\dfrac{d}{dx}\sinh^{-1}x = \dfrac{1}{\sqrt{1+x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\sinh x$ and differentiate both sides.

$$\begin{aligned}
\sinh\left(\sinh^{-1}x\right) &= x \\
\cosh\left(\sinh^{-1}x\right)\frac{d}{dx}\left(\sinh^{-1}x\right) &= 1 \\
\frac{d}{dx}\left(\sinh^{-1}x\right) &= \frac{1}{\cosh\left(\sinh^{-1}x\right)}
\end{aligned}$$

Recall that for all y, $\cosh^2 y - \sinh^2 y = 1$. Then $\cosh y = \pm\sqrt{1+\sinh^2 y}$. Since $\cosh y$ is always positive, the negative solution is easily ruled out. Thus

$$\frac{d}{dx}\left(\sinh^{-1}x\right) = \frac{1}{\cosh\left(\sinh^{-1}x\right)} = \frac{1}{\sqrt{1+\sinh^2\left(\sinh^{-1}x\right)}} = \frac{1}{\sqrt{1+x^2}}$$

Theorem: $\boxed{\dfrac{d}{dx}\cosh^{-1}x = \dfrac{1}{\sqrt{x^2-1}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\cosh x$ and differentiate both sides. Recall that the domain of $\cosh^{-1}x$ is $[1,\infty)$ (since it is the range of $\cosh x$). On that domain, $\sinh x$ is always positive.

$$\begin{aligned}
\cosh\left(\cosh^{-1}x\right) &= x \\
\sinh\left(\cosh^{-1}x\right)\frac{d}{dx}\left(\cosh^{-1}x\right) &= 1 \\
\frac{d}{dx}\left(\cosh^{-1}x\right) &= \frac{1}{\sinh\left(\cosh^{-1}x\right)}
\end{aligned}$$

Recall that for all y, $\cosh^2 y - \sinh^2 y = 1$. Then $\sinh y = \pm\sqrt{\cosh^2 y - 1}$. When $y \geq 1$, $\sinh y$ is positive, and so the negative solution is ruled out. Thus

$$\frac{d}{dx}\left(\cosh^{-1}x\right) = \frac{1}{\sinh\left(\cosh^{-1}x\right)} = \frac{1}{\sqrt{\cosh^2\left(\cosh^{-1}x\right)-1}} = \frac{1}{\sqrt{x^2-1}}$$

Theorem: $\boxed{\dfrac{d}{dx}\tanh^{-1}x = \dfrac{1}{1-x^2}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\tanh x$ and differentiate both sides. Recall that the domain of $\tanh^{-1}x$ is $(-1,1)$.

$$\begin{aligned}
\tanh\left(\tanh^{-1}x\right) &= x \\
\operatorname{sech}^2\left(\tanh^{-1}x\right)\frac{d}{dx}\left(\tanh^{-1}x\right) &= 1 \\
\frac{d}{dx}\left(\tanh^{-1}x\right) &= \frac{1}{\operatorname{sech}^2\left(\tanh^{-1}x\right)}
\end{aligned}$$

Recall that for all $\tanh^2 y = 1-\operatorname{sech}^2 y$. Thus

$$\frac{d}{dx}\left(\tanh^{-1}x\right) = \frac{1}{\operatorname{sech}^2\left(\tanh^{-1}x\right)} = \frac{1}{1-\tanh^2\left(\tanh^{-1}x\right)} = \frac{1}{1-x^2}$$

Theorem: $\boxed{\dfrac{d}{dx}\coth^{-1}x = \dfrac{1}{1-x^2}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\tanh x$ and differentiate both sides. Recall that the domain of $\coth^{-1}x$ is $(-\infty,-1)\cup(1,\infty)$.

$$\begin{aligned}
\coth\left(\coth^{-1}x\right) &= x \\
-\operatorname{csch}^2\left(\coth^{-1}x\right)\frac{d}{dx}\left(\coth^{-1}x\right) &= 1 \\
\frac{d}{dx}\left(\coth^{-1}x\right) &= -\frac{1}{\operatorname{csch}^2\left(\coth^{-1}x\right)}
\end{aligned}$$

Recall that for all $\coth^2 y = 1+\operatorname{csch}^2 y$. Thus $\operatorname{csch}^2 y = \coth^2 y - 1$ and so

$$\frac{d}{dx}\left(\coth^{-1}x\right) = -\frac{1}{\operatorname{csch}^2\left(\coth^{-1}x\right)} = -\frac{1}{\coth^2\left(\coth^{-1}x\right)-1} = -\frac{1}{x^2-1} = \frac{1}{1-x^2}$$

There is another way to prove this. Recall that $\coth^{-1}x = \tanh^{-1}\left(\dfrac{1}{x}\right)$

$$\frac{d}{dx}\coth^{-1}x = \frac{d}{dx}\tanh^{-1}\left(\frac{1}{x}\right) = \frac{1}{1-\left(\frac{1}{x}\right)^2}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right) = \frac{1}{1-\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) = \frac{1}{1-\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{-1}{x^2-1} = \frac{1}{1-x^2}$$

Theorem: $\boxed{\dfrac{d}{dx}\operatorname{sech}^{-1} x = \dfrac{-1}{x\sqrt{1-x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\operatorname{sech} x$ and differentiate both sides. Recall that the domain of $\operatorname{sech}^{-1} x$ is $[0,1)$ and range $[0,\infty)$.

$$\begin{aligned}
\operatorname{sech}\left(\operatorname{sech}^{-1} x\right) &= x \\
-\operatorname{sech}\left(\operatorname{sech}^{-1} x\right)\cdot\tanh\left(\operatorname{sech}^{-1} x\right)\cdot\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) &= 1 \\
\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) &= -\frac{1}{\operatorname{sech}\left(\operatorname{sech}^{-1} x\right)\cdot\tanh\left(\operatorname{sech}^{-1} x\right)} = -\frac{1}{x\tan\left(\operatorname{sech}^{-1} x\right)}
\end{aligned}$$

Recall that for all $\tanh^2 y = 1-\operatorname{sech}^2 y$. Thus $\tanh y = \pm\sqrt{1-\operatorname{sech}^2 y}$. In this case, $\tanh\left(\operatorname{sech}^{-1} x\right)$ must be positive and so $\tanh y = \sqrt{1-\operatorname{sech}^2 y}$. Thus

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-\operatorname{sech}^2\left(\operatorname{sech}^{-1} x\right)}} = -\frac{1}{x\sqrt{1-x^2}}$$

There is another way to prove this. Recall that $\operatorname{sech}^{-1} x = \cosh^{-1}\left(\dfrac{1}{x}\right)$

$$\begin{aligned}
\frac{d}{dx}\operatorname{sech}^{-1} x &= \frac{d}{dx}\cosh^{-1}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{\frac{1}{x^2}-1}}\left(-\frac{1}{x^2}\right) = \frac{-1}{x^2\sqrt{\frac{1}{x^2}-1}} \\
&= \frac{-1}{x\sqrt{x^2\left(\frac{1}{x^2}-1\right)}} = \frac{-1}{x\sqrt{1-x^2}}
\end{aligned}$$

Theorem: $\boxed{\dfrac{d}{dx}\operatorname{csch}^{-1} x = -\dfrac{1}{|x|\sqrt{1+x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f\left(f^{-1}(x)\right) = x$$

We will state that for $\operatorname{csch} x$ and differentiate both sides. Recall that the domain of $\operatorname{csch}^{-1} x$ is $(-\infty,0)\cup(0,\infty)$ and range $(-\infty,0)\cup(0,\infty)$.

$$\begin{aligned}
\operatorname{csch}\left(\operatorname{csch}^{-1} x\right) &= x \\
-\operatorname{csch}\left(\operatorname{csch}^{-1} x\right)\cdot\coth\left(\operatorname{csch}^{-1} x\right)\cdot\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) &= 1 \\
\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) &= -\frac{1}{\operatorname{csch}\left(\operatorname{csch}^{-1} x\right)\cdot\coth\left(\operatorname{csch}^{-1} x\right)} = -\frac{1}{x\coth\left(\operatorname{csch}^{-1} x\right)}
\end{aligned}$$

Recall that for all $\coth^2 y = 1 + \operatorname{csch}^2 y$. Thus $\coth y = \pm\sqrt{1 + \operatorname{csch}^2 y}$.

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{x\coth\left(\operatorname{csch}^{-1} x\right)} = -\frac{1}{\pm x\sqrt{1 + \operatorname{csch}^2\left(\operatorname{csch}^{-1} x\right)}} = \pm\frac{1}{x\sqrt{1+x^2}}$$

looking at the graph of $\operatorname{csch}^{-1} x$, it is always decreasing and so its derivative must be negative. Thus

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1+x^2}}$$

There is another way to prove this. Recall that $\operatorname{csch}^{-1} x = \sinh^{-1}\left(\frac{1}{x}\right)$

$$\begin{aligned}\frac{d}{dx}\operatorname{csch}^{-1} x &= \frac{d}{dx}\sinh^{-1}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{1+\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right) = \frac{-1}{x^2\sqrt{1+\frac{1}{x^2}}}\\ &= \frac{-1}{|x|\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{-1}{|x|\sqrt{x^2+1}}\end{aligned}$$

Formulas for inverse functions

A more direct formula for $y = \sinh^{-1} x$ exists.

Theorem: $\boxed{\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)}$.

Proof: Consider $y = \sinh^{-1} x$, same as $\sinh y = x$. We will solve this for y.

$$\begin{aligned} x &= \sinh y \\ x &= \frac{1}{2}\left(e^y - e^{-y}\right) \\ x &= \frac{1}{2}\left(e^y - \frac{1}{e^y}\right) \qquad \text{multiply by } e^y \\ xe^y &= \frac{1}{2}\left(e^y\right)^2 - \frac{1}{2} \end{aligned}$$

This is quadratic in e^y. We will solve for e^y and then for y.

$$\begin{aligned} xe^y &= \frac{1}{2}\left(e^y\right)^2 - \frac{1}{2} \\ 0 &= \frac{1}{2}\left(e^y\right)^2 - xe^y - \frac{1}{2} \end{aligned}$$

Applying the quadratic formula,

$$e^y_{1,2} = \frac{x \pm \sqrt{x^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}}{2\left(\frac{1}{2}\right)} = \frac{x \pm \sqrt{x^2+1}}{1} = x \pm \sqrt{x^2+1}$$

Since $\sqrt{x^2+1}$ has a greater absolute value than x, $x - \sqrt{x^2+1}$ is always negative and $x + \sqrt{x^2+1}$ is always positive. Because e^y is positive for all real y, we discard the negative solution and solve for y.

$$\begin{aligned} e^y &= x + \sqrt{x^2+1} \\ y &= \ln\left(x + \sqrt{x^2+1}\right) \end{aligned}$$

We can differentiate $\sinh^{-1} x$ using this form of it:

$$\begin{aligned} \frac{d}{dx}\sinh^{-1} x &= \frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}}(2x)\right) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Similar computation yields for $\boxed{\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)}$.

Now for $\tanh^{-1} x$: Recall that this function has domain $(-1, 1)$.

$$\begin{aligned}
y &= \tanh^{-1} x \\
\tanh y &= x \\
x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
x\left(e^y + e^{-y}\right) &= e^y - e^{-y} \\
xe^y + x\frac{1}{e^y} &= e^y - \frac{1}{e^y} \qquad \text{multiply by } e^y \\
x\left(e^y\right)^2 + x &= \left(e^y\right)^2 - 1 \\
x + 1 &= \left(e^y\right)^2 - x\left(e^y\right)^2 \\
x + 1 &= \left(e^y\right)^2 (1 - x) \\
\frac{1+x}{1-x} &= \left(e^y\right)^2 \\
e^y &= \pm\sqrt{\frac{1+x}{1-x}}
\end{aligned}$$

Since e^y is always positive, we discard the negative solution and solve for y.

$$\begin{aligned}
e^y &= \sqrt{\frac{1+x}{1-x}} \\
y &= \ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right)
\end{aligned}$$

We can differentiate $\tanh^{-1} x$ using this form of it:

$$\begin{aligned}
\frac{d}{dx}\tanh^{-1} x &= \frac{d}{dx}\left[\frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right)\right] = \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{1-x}(-1)\right) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \\
&= \frac{1}{2}\left(\frac{1-x}{(1+x)(1-x)} + \frac{1+x}{(1-x)(1+x)}\right) = \frac{1}{2}\left(\frac{1-x+1+x}{1-x^2}\right) = \frac{1}{2}\left(\frac{2}{1-x^2}\right) = \frac{1}{1-x^2}
\end{aligned}$$

Similar formulas exist for all inverse hyperbolic inverse functions:

$$\begin{aligned}
\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) & x &\in \mathbb{R} \\
\cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) & |x| &\geq 1 \\
\tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) & |x| &< 1 \\
\operatorname{sech}^{-1} x &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) & 0 &< x \leq 1 \\
\operatorname{csch}^{-1} x &= \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) & x &\neq 0 \\
\coth^{-1} x &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = \frac{1}{2}\left(\ln(x+1) - \ln(x-1)\right) & |x| &> 1
\end{aligned}$$

Formulas

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1\\ \tanh^2 x &= 1 - \operatorname{sech}^2 x\\ \coth^2 x &= 1 + \operatorname{csch}^2 x\end{aligned}$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\begin{aligned}\cosh 2x &= \cosh^2 x + \sinh^2 x\\ &= 1 + 2\sinh^2 x\\ &= 2\cosh^2 x - 1\end{aligned}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \int \cosh x \; dx = \sinh x + C$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \int \sinh x \; dx = \cosh x + C$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \int \operatorname{sech}^2 x \; dx = \tanh x + C$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \qquad \int \operatorname{csch}^2 x \; dx = -\coth x + C$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad \int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x \qquad \int \operatorname{csch} x \coth x \; dx = -\operatorname{csch} x + C$$

$$\begin{aligned}\int \tanh x \; dx &= \ln(\cosh x) + C\\ \int \coth x \; dx &= \ln|\sinh x| + C\end{aligned}$$

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}} \qquad \int \frac{1}{\sqrt{x^2+1}}\; dx = \sinh^{-1} x + C$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} \quad x > 1 \qquad \int \frac{1}{\sqrt{x^2-1}}\; dx = \cosh^{-1} x + C \quad x > 1$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} \quad |x| < 1 \qquad \int \frac{1}{1-x^2}\; dx = \tanh^{-1} x + C \quad |x| < 1$$

$$\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2} \quad |x| > 1 \qquad \int \frac{1}{1-x^2}\; dx = \coth^{-1} x + C \quad |x| > 1$$

$$\frac{d}{dx}\operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}} \quad 0 < x < 1 \qquad \int \frac{1}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + C \quad 0 < x < 1$$

$$\frac{d}{dx}\operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}} \quad x \neq 0 \qquad \int \frac{1}{|x|\sqrt{1+x^2}} = -\operatorname{csch}^{-1} x + C \quad x \neq 0$$

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