Recall the definitions of the **hyperbolic cosine** and **hyperbolic sine** functions as

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x}$$

and also recall that

$$\cosh^2 x - \sinh^2 x = 1 \quad \text{for all real number } x.$$

Also note that when solving for one in terms of the other, $\cosh^2 x$ and $\sinh^2 x$ behave slightly differently. Algebraically, $\cosh^2 x = \pm\sqrt{1+\sinh^2 x}$ and $\sinh^2 x = \pm\sqrt{\cosh^2 x - 1}$. However, $\cosh x$ is always positive (as a matter of fact, always at least 1) and so

$$\cosh x = \sqrt{1+\sinh^2 x}$$

while $\sinh x$ is negative for negative x and so

$$\sinh x = \pm\sqrt{\cosh^2 x - 1} = \begin{cases} \sqrt{\cosh^2 x - 1} & \text{if} \quad x \geq 0 \\ -\sqrt{\cosh^2 x - 1} & \text{if} \quad x < 0 \end{cases}$$

It is easy to verify that $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$. Therefore,

Theorem 1: $\int \sinh x dx = \cosh x + C$ and $\int \cosh x dx = \sinh x + C$

Theorem 2: $\int \tanh x dx = \ln(\cosh x) + C$ and $\int \coth x dx = \ln|\sinh x| + C$

proof: We will compute $\int \tanh x dx$.
We will use substitution. Let $u = \cosh x$. Then $du = \sinh x dx$ and so

$$\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x}(\sinh x dx) = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C = \ln(\cosh x) + C$$

The last step is because $\cosh x$ is always positive. The other integral, $\int \coth x dx$ goes very similarly, using the substitution $u = \sinh x$.

Theorem 3: $\int \operatorname{sech} x dx = 2\tan^{-1}(e^x) + C$

proof:

$$\begin{aligned}\int \operatorname{sech} x dx &= \int \frac{1}{\cosh x} dx = \int \frac{2}{e^x + e^{-x}} dx = \int \frac{2}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} dx = \int \frac{2e^x}{(e^x)^2 + e^{-x}(e^x)} dx \\ &= \int \frac{2e^x}{(e^x)^2 + e^{-x+x}} dx = \int \frac{2e^x}{(e^x)^2 + e^0} dx = \int \frac{2e^x}{(e^x)^2 + 1} dx\end{aligned}$$

Let $u = e^x$. Then $du = e^x dx$.

$$\int \frac{2e^x}{(e^x)^2+1}dx = \int \frac{2}{(e^x)^2+1}(e^x dx) = 2\int \frac{1}{u^2+1}du = 2\tan^{-1} u + C = 2\tan^{-1}(e^x) + C$$

There are other methods of integrating this function. For 4 additional methods, click here.

Theorem 4: $\displaystyle\int \operatorname{csch} x = \ln\left|\frac{e^x-1}{e^x+1}\right| + C$

proof:

$$\begin{aligned}\int \operatorname{csch} x dx &= \int \frac{1}{\sinh x}dx = \int \frac{2}{e^x - e^{-x}}dx = \int \frac{2}{e^x - e^{-x}} \cdot \frac{e^x}{e^x}dx = \int \frac{2}{e^x - e^{-x}} \cdot \frac{e^x}{e^x}dx \\ &= \int \frac{2e^x}{(e^x)^2 - e^{-x}(e^x)}dx = \int \frac{2e^x}{(e^x)^2-1}dx\end{aligned}$$

Let $u = e^x$. Then $du = e^x dx$.

$$\int \frac{2e^x}{(e^x)^2-1}dx = \int \frac{2}{(e^x)^2-1}(e^x dx) = \int \frac{2}{u^2-1}du = \int \frac{2}{(u+1)(u-1)}du$$

We will proceed using partial fractions

$$\begin{aligned}\frac{2}{(u+1)(u-1)} &= \frac{A}{u-1} + \frac{B}{u+1} \\ 2 &= A(u+1) + B(u-1)\end{aligned}$$

Let $u = 1$

$$2 = 2A \qquad \Longrightarrow \qquad A = 1$$

Let $u = -1$

$$2 = B(-2) \qquad \Longrightarrow \qquad B = -1$$

So

$$\begin{aligned}\int \frac{2}{(u+1)(u-1)}du &= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \int \frac{1}{u-1}du - \int \frac{1}{u+1}du = \ln|u-1| - \ln|u+1| + C \\ &= \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{e^x-1}{e^x+1}\right| + C\end{aligned}$$

This answer has many different forms. We can prove that $\ln\left|\dfrac{e^x-1}{e^x+1}\right| + C$, $\ln\left|\tanh\left(\dfrac{x}{2}\right)\right| + C$, and

$\ln|\coth x - \operatorname{csch} x| + C$ are all expressing the same integral. For additional methods of integrating this function, click here.

Inverse Functions

Theorem 5: $\displaystyle\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \sqrt{x^2+1} + C$

proof: We will first need to compute the derivative of $\sinh^{-1} x$. This computation is in the previous handout but we will compute it again here using implicit differentiation. Recall again that $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \qquad \text{differentiate both sides} \\ \cosh y \cdot y' &= 1 \\ y' &= \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

We are now ready to integrate $\sinh^{-1} x$. We will integrate by parts, using the formula

$$\int u\,dv = uv - \int v\,du$$

Let $u = \sinh^{-1} x$ and $dv = dx$. Then $du = \dfrac{1}{\sqrt{1+x^2}}dx$ and $v = x$ and the statement $\displaystyle\int u\,dv = uv - \int v\,du$ becomes

$$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int x\frac{1}{\sqrt{x^2+1}}dx$$

We will evaluate the second integral using substitution. Let $u = x^2+1$ and so $du = 2x\,dx$

$$\int x\frac{1}{\sqrt{x^2+1}}dx = \int \frac{\frac{1}{2}(2x\,dx)}{\sqrt{x^2+1}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{x^2+1} + C$$

and so the integral is

$$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int x\frac{1}{\sqrt{x^2+1}}dx = x\sinh^{-1} x - \sqrt{x^2+1} + C$$

We check our answer via differentiation:

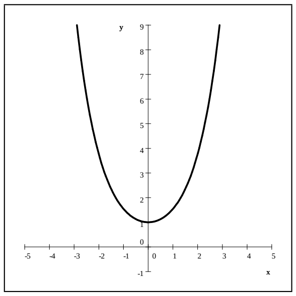
$$\frac{d}{dx}\left(x\sinh^{-1} x - \sqrt{x^2+1}\right) = 1\cdot\sinh^{-1} x + x\cdot\frac{1}{\sqrt{1+x^2}} - \frac{1}{2\sqrt{x^2+1}}(2x) = \sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{x^2+1}} = \sinh^{-1} x$$

Theorem 6: $\displaystyle\int \cosh^{-1} x\,dx = x\cosh^{-1} x - \sqrt{x^2-1} + C$

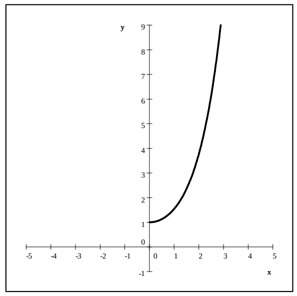
proof: We will first need to compute the derivative of $\cosh^{-1} x$. Recall that $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned} y &= \cosh^{-1} x \\ \cosh y &= x \qquad \text{differentiate both sides} \\ \sinh y \cdot y' &= 1 \\ y' &= \frac{1}{\sinh y} = \frac{1}{\pm\sqrt{\sinh^2 y - 1}} = \frac{1}{\pm\sqrt{x^2-1}} \end{aligned}$$

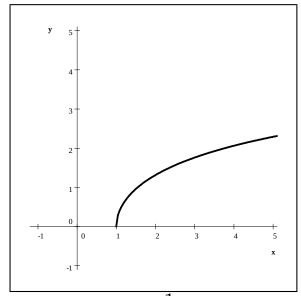
We will need to figure out the sign in the derivative. Recall that $\cosh x$ is not one-to-one, so its domain had to be restricted for the definition of inverse function. We restricted its domain to be $[0,\infty)$ and over that interval, $\cosh x$ is increasing.



$\cosh x$



restricted $\cosh x$



$\cosh^{-1} x$

Since the inverse of an increasing function is also increasing, $\cosh^{-1} x$ is an increasing function. Then its derivative in non-negaitive. Thus, $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}$.

We are now ready to integrate $\cosh^{-1} x$. We will integrate by parts, using the formula

$$\int u dv = uv - \int v du$$

Let $u = \cosh^{-1} x$ and $dv = dx$. Then $du = \dfrac{1}{\sqrt{x^2-1}} dx$ and $v = x$ and the statement $\int u dv = uv - \int v du$ becomes

$$\int \cosh^{-1} x dx = x \cosh^{-1} x - \int x \frac{1}{\sqrt{x^2-1}} dx$$

We will evaluate the second integral using substitution. Let $u = x^2 - 1$ and so $du = 2x dx$

$$\int x \frac{1}{\sqrt{x^2-1}} dx = \int \frac{\frac{1}{2}(2x dx)}{\sqrt{x^2-1}} = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2} du = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{x^2-1} + C$$

and so the integral is

$$\int \cosh^{-1} x dx = x \cosh^{-1} x - \int x \frac{1}{\sqrt{x^2-1}} dx = x \cosh^{-1} x - \sqrt{x^2-1} + C$$

We will leave it to the reader to check our answer via differentiation.

Theorem 7: $\displaystyle\int \tanh^{-1} x dx = x \tanh^{-1} x + \frac{1}{2} \ln\left(1 - x^2\right) + C$

proof: We will first need to compute the derivative of $\tanh^{-1} x$. Recall that $\cosh^2 x - \sinh^2 x = 1$. When we divide both sides by $\cosh^2 x$, we get $\dfrac{\cosh^2 x}{\cosh^2 x} - \dfrac{\sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$ and so $1 - \tanh^2 x = \operatorname{sech}^2 x$. Recall that $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x = 1 - \tanh^2 x$.

$$\begin{aligned} y &= \tanh^{-1} x \\ \tanh y &= x && \text{differentiate both sides} \\ \operatorname{sech}^2 y \cdot y' &= 1 \\ y' &= \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 x} = \frac{1}{1 - x^2} \end{aligned}$$

We are now ready to integrate $\tanh^{-1} x$. We will integrate by parts, using the formula

$$\int u dv = uv - \int v du$$

Let $u = \tanh^{-1} x$ and $dv = dx$. Then $du = \dfrac{1}{1-x^2}dx$ and $v = x$ and the statement $\displaystyle\int udv = uv - \int vdu$ becomes

$$\int \tanh^{-1} x dx = x\tanh^{-1} x - \int x\frac{1}{1-x^2}dx$$

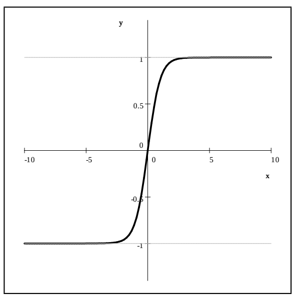
We will evaluate the second integral using substitution (although partial fractions would also work). Let $u = 1-x^2$ and so $du = -2xdx$

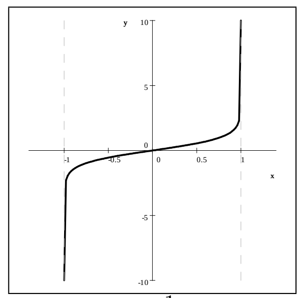
$$\int x\frac{1}{1-x^2}dx = \int \frac{-\frac{1}{2}(-2xdx)}{1-x^2} = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln\left|1-x^2\right| + C$$

and so the entire integral is

$$\int \tanh^{-1} x dx = x\tanh^{-1} x - \int x\frac{1}{1-x^2}dx = x\tanh^{-1} - \left(-\frac{1}{2}\ln\left|1-x^2\right|\right) + C = x\tanh^{-1} x + \frac{1}{2}\ln\left|1-x^2\right| + C$$

We should note that this result can be simplified as $\ln\left|1-x^2\right| = \ln\left(1-x^2\right)$. This is because in case of $\tanh^{-1} x$, the domain is $(-1,1)$.


$\tanh x$

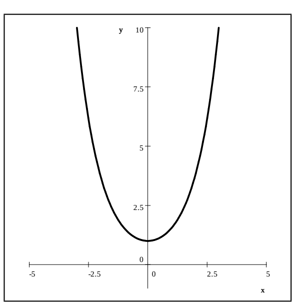

$\tanh^{-1} x$

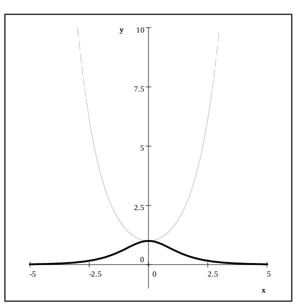
So the final answer is $\displaystyle\int \tanh^{-1} x dx = x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right) + C$

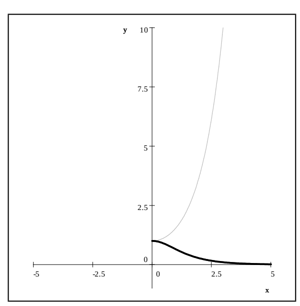
We will leave it to the reader to check our answer via differentiation.

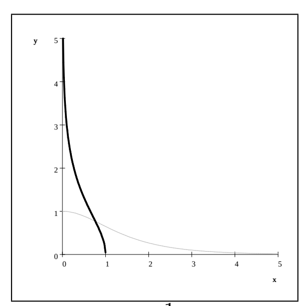
Theorem 8: $\displaystyle\int \operatorname{sech}^{-1} x dx = x\operatorname{sech}^{-1} x + \sin^{-1} x + C$

proof: We will first need to compute the derivative of $\operatorname{sech}^{-1} x$. Recall that $\cosh^2 x - \sinh^2 x = 1$. When we divide both sides by $\cosh^2 x$, we get $\dfrac{\cosh^2 x}{\cosh^2 x} - \dfrac{\sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$ and so $1 - \tanh^2 x = \operatorname{sech}^2 x$. Recall that $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$. Also, we will need to know certain properties of the function. Let us graph $\operatorname{sech}^{-1} x$.


$\cosh x$


$\operatorname{sech} x$


$\operatorname{sech} x$ restricted


$\operatorname{sech}^{-1} x$

Before the computations, it is useful to know that the domain of $\operatorname{sech}^{-1} x$ is $(0,1]$ and that it is a decreasing function.

$$\begin{aligned} y &= \operatorname{sech}^{-1} x \\ \operatorname{sech} y &= x \qquad \text{differentiate both sides} \\ -\operatorname{sech} y \tanh y \cdot y' &= 1 \\ y' &= \frac{1}{-\operatorname{sech} y \tanh y} = \frac{-1}{\operatorname{sech} y \left(\pm\sqrt{1-\operatorname{sech}^2 y}\right)} = \frac{-1}{x\left(\pm\sqrt{1-x^2}\right)} \end{aligned}$$

Since x is always positive (recall the domain of $\operatorname{sech}^{-1} x$ is $(0,1]$) and it is a decreasing function, the derivative is $\dfrac{-1}{x\sqrt{1-x^2}}$. We are now ready to integrate $\tanh^{-1} x$. We will integrate by parts, using the formula

$$\int u dv = uv - \int v du$$

Let $u = \operatorname{sech}^{-1} x$ and $dv = dx$. Then $du = \dfrac{-1}{x\sqrt{1-x^2}}dx$ and $v = x$ and the statement $\int u dv = uv - \int v du$ becomes

$$\int \operatorname{sech}^{-1} x dx = x \operatorname{sech}^{-1} x - \int x \frac{-1}{x\sqrt{1-x^2}} dx = x \operatorname{sech}^{-1} x + \int \frac{1}{\sqrt{1-x^2}} dx = x \operatorname{sech}^{-1} x + \sin^{-1} x + C$$

We check our answer via differentiation:

$$\frac{d}{dx}\left(x \operatorname{sech}^{-1} x + \sin^{-1} x\right) = \operatorname{sech}^{-1} x + x\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{1}{\sqrt{1-x^2}} = \operatorname{sech}^{-1} x - \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = \operatorname{sech}^{-1} x$$

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