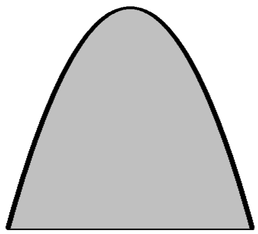
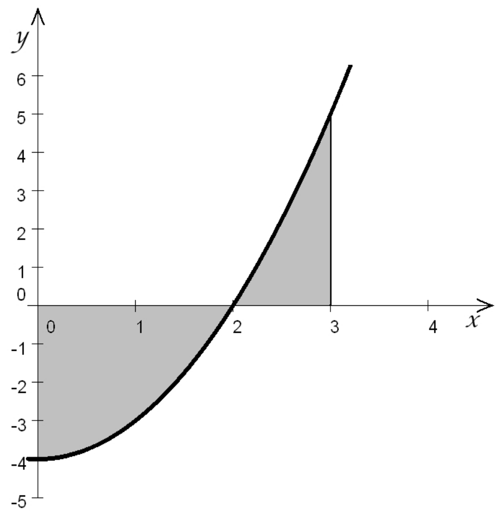
Sample Problems

1. We need to build a door in the shape of a parabola. The door is to be of width of 6 feet and height of 9 feet. What is the surface area of this door?



2. Find the shaded area shown on the picture below. The function graphed there is $f(x) = x^2 - 4$.



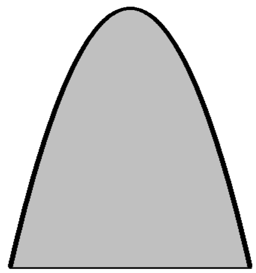
3. Find the area enclosed by the graphs of $f(x) = x^2 + 1$ and $g(x) = 2x + 4$.

4. Find the area of the region determined by the graphs of $x = y^2 - 2$ and $y = x$.

5. Find the average value of $f(x) = x^2$ on the interval $[0, 6]$.

Practice Problems

1. We need to build a door in the shape of a parabola. The door is to be of width of 4 feet and height of 8 feet. What is the surface area of this door?



2. Compute the area of the region determined by the $x-$axis and the graph of $y = -x^2 + 16$ between $x = 0$ and $x = 7$.

3. Find the area enclosed by the graphs of

   a) $f(x) = -x^2 + 9$ and $g(x) = -2x + 1$

   b) $f(x) = x^2 + 6x - 7$ and $g(x) = 5x + 5$

4. Compute the area of the region determined by the graphs of $x = y^2 - 9$ and $y = \frac{1}{4}x + 1$

5. Find the average value of each of the following functions on the interval indicated.

   a) $f(x) = x^3$ on $[0, 1]$

   b) $f(x) = \sin x$ on $[0, \pi]$

   c) $f(x) = 2^x$ on $[1, 4]$

   d) $f(x) = \frac{1}{x}$ on $[1, 10]$

   e) $f(x) = e^{-2x}$ on $[0, 2]$

   f) $f(x) = \frac{1}{\sqrt{x^2 - 1}}$ on $[1, 5]$

Sample Problems - Answers

1.) 36 2.) $\dfrac{23}{3}$ 3.) $\dfrac{32}{3}$ 4.) $\dfrac{9}{2}$ 5.) 12

Practice Problems - Answers

1.) $\dfrac{64}{3}$ 2.) $\dfrac{263}{3}$ 3.) a) 36 b) $\dfrac{343}{6}$ 4.) 36

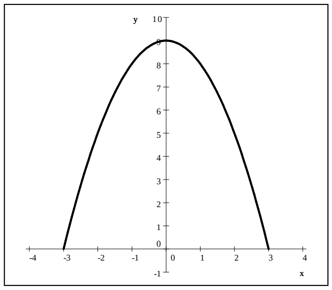
5.) a) $\dfrac{1}{4}$ b) $\dfrac{2}{\pi}$ c) $\dfrac{14}{3\ln 2} \approx 6.7326$ d) $\dfrac{1}{9}\ln 10 \approx 0.25584$ e) $\dfrac{1}{4} - \dfrac{1}{4e^4} \approx 0.24542$

f) $\dfrac{1}{4}\cosh^{-1} 5 \approx 0.57311$

Sample Problems - Solutions

1. We need to build a door in the shape of a parabola. The door is to be of width of 6 feet and height of 9 feet. What is the surface area of this door?

   Solution: Let us place a coordinate system on the door so that the origin falls into the center of the base of the door. then the base of the door falls between $(-3, 0)$ and $(3, 0)$.

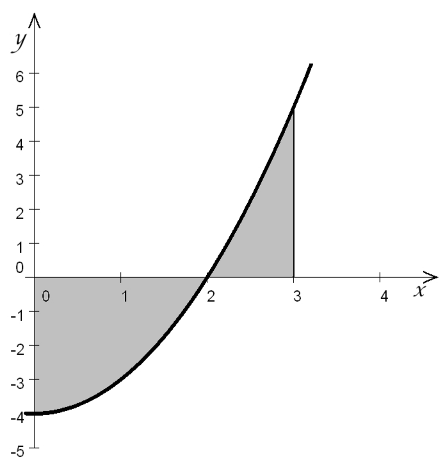
   If the parabola is to pass through those two points, its equation must be $y = a(x+3)(x-3)$ where clearly a is negative. We will find the value of a using the height of the parabola. $(0, 9)$ is also a point of the parabola. $9 = a(0+3)(0-3)$ gives us the vaue of $a = -1$. The area of the door is the definite integral

$$\int_{-3}^{3} \left(-x^2+9\right) dx = 2\int_{0}^{3} \left(-x^2+9\right) dx = 2\left(-\frac{x^3}{3}+9x\Big|_0^3\right) = 2\left[\left(-\frac{3^3}{3}+9\cdot 3\right)-\left(-\frac{0^3}{3}+9\cdot 0\right)\right] = 2\cdot 18 = \boxed{36}$$

2. Find the shaded area shown on the picture below. The function graphed there is $f(x) = x^2 - 4$.

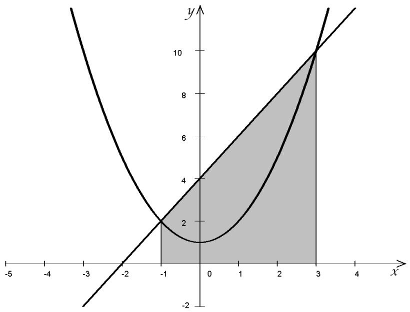
   Solution: We need to be careful with the signs. The function is negative on $(0, 2)$ and positive on $(2, 3)$. Consequently, the total area can not be found as the definite integral between 0 and 3. We need to compute the positive and negative areas separately:

$$\begin{aligned} A &= A_1 + A_2 = -\int_0^2 \left(x^2-4\right) dx + \int_2^3 \left(x^2-4\right) dx = -\frac{x^3}{3}+4x\Big|_0^2 + \frac{x^3}{3}-4x\Big|_2^3 \\ &= \left(-\frac{2^3}{3}+4\cdot 2\right)-\left(-\frac{0^3}{3}+4\cdot 0\right)+\left(\frac{3^3}{3}-4\cdot 3\right)-\left(\frac{2^3}{3}-4\cdot 2\right) \\ &= \left(-\frac{8}{3}+8\right)+\left(\frac{27}{3}-12\right)-\left(\frac{8}{3}-8\right) = \frac{16}{3}+(-3)-\left(-\frac{16}{3}\right) = \frac{32}{3}-\frac{9}{3} = \boxed{\frac{23}{3}} \end{aligned}$$

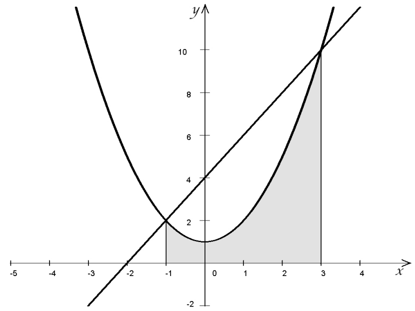


3. Find the area enclosed by the graphs of $f(x) = x^2 + 1$ and $g(x) = 2x + 4$.

Solution: We first solve the system $\begin{cases} y = x^2 + 1 \\ y = 2x + 4 \end{cases}$ to find where the graphs intersect. Using substitution, we obtain $(-1, 2)$ and $(3, 10)$. In this region, the line is above the parabola.



A_1 - area under the line

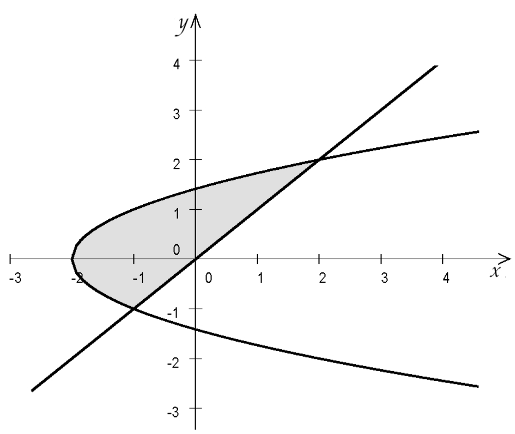


A_2 - area under the parabola

As the pictures above show, the area between the two graphs can be computed as a difference between two areas; we need to subtract the area under the parabola from the area under the line.

$$\begin{aligned} A &= A_1 - A_2 = \int_{-1}^{3} 2x + 4 \, dx - \int_{-1}^{3} x^2 + 1 \, dx = \int_{-1}^{3} (2x+4) - \left(x^2 + 1\right) \, dx \\ &= \int_{-1}^{3} -x^2 + 2x + 3 \, dx = -\frac{x^3}{3} + x^2 + 3x \Big|_{-1}^{3} = \left(-\frac{3^3}{3} + 3^2 + 3 \cdot 3\right) - \left(-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right) \\ &= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = \boxed{\frac{32}{3}} \end{aligned}$$

4. Find the area of the region determined by the graphs of $x = y^2 - 2$ and $y = x$.



We solve for y in $x = y^2 - 2$ and obtain $y = \pm\sqrt{x+2}$. Betwen -2 and -1, the region is that between $y = \sqrt{x+2}$ and $y = \sqrt{x+2}$. Between -1 and 2, the region is that between $y = \sqrt{x+2}$ and $y = x$. In both cases, we compute the area as the difference between definite integrals as before.

$$\begin{aligned}
A &= \int_{-2}^{-1} \sqrt{x+2} - \left(-\sqrt{x+2}\right)\ dx + \int_{-1}^{2} \sqrt{x+2} - x\ dx = \int_{-2}^{-1} 2\sqrt{x+2}\ dx + \int_{-1}^{2} \sqrt{x+2} - x\ dx \\
&= 2\left(\frac{2}{3}\right)(x+2)^{3/2}\Big|_{-2}^{-1} + \left(\frac{2}{3}(x+2)^{3/2} - \frac{x^2}{2}\right)\Big|_{-1}^{2} \\
&= \frac{4}{3}(-1+2)^{3/2} - \frac{4}{3}(-2+2)^{3/2} + \left(\frac{2}{3}(2+2)^{3/2} - \frac{2^2}{2}\right) - \left(\frac{2}{3}(-1+2)^{3/2} - \frac{(-1)^2}{2}\right) \\
&= \frac{4}{3}(1-0) + \left(\frac{2}{3}\cdot 4^{3/2} - 2\right) - \left(\frac{2}{3}\cdot 1^{3/2} - \frac{1}{2}\right) = \frac{4}{3} + \left(\frac{2}{3}\cdot 8 - 2\right) - \left(\frac{2}{3} - \frac{1}{2}\right) \\
&= \frac{4}{3} + \left(\frac{16}{3} - 2\right) - \left(\frac{2}{3} - \frac{1}{2}\right) = \frac{4}{3} + \frac{10}{3} - \frac{1}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}
\end{aligned}$$

Note: there are other ways to solve this problem. One option is to invert the equations of the graphs and reduce the question to one similar to the previous one. Another option is to integrate in terms of y as it ranges from -1 to 2.

$$A = \int_{-1}^{2} (x_1 - x_2)\, dy = \int_{-1}^{2} \left(y - \left(y^2 - 2\right)\right) dy = \int_{-1}^{2} -y^2 + y - 2\ dy = \frac{9}{2}$$

5. Find the average value of $f(x) = x^2$ on the interval $[0, 6]$.

   Solution: The geometric meaning of the average value of an integrable function is the height of the rectangle with the same area as the definite integral. Suppose that f is integrable on $[a, b]$.

$$\begin{aligned}
A_{\text{rectangle}} &= A_{\text{under graph}} \\
H(b-a) &= \int_a^b f(x)\ dx \qquad \Longrightarrow \qquad H = \frac{1}{b-a}\int_a^b f(x)\ dx
\end{aligned}$$

   In this case,

$$Av = \frac{1}{6-0}\int_0^6 x^2\ dx = \frac{1}{6}\left(\frac{x^3}{3}\Big|_0^6\right) = \frac{1}{6}\left(\frac{6^3}{3} - \frac{0^3}{3}\right) = \frac{1}{6}\left(\frac{216}{3}\right) = \boxed{12}$$

