Sample Problems

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = 2(x-1)^{3/2}$ on $[1,5]$
2. $f(x) = \dfrac{2}{3}\left(x^2+1\right)^{3/2}$ on $[1,4]$
3. $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on $[1,3]$
4. $f(x) = \ln(\cos x)$ on $\left[0, \dfrac{\pi}{4}\right]$

Practice Problems

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = \cosh x$ on $[0,1]$
2. $f(x) = \sqrt{4-x^2}$ on $[-2,2]$
3. $f(x) = x^2 - \dfrac{1}{8}\ln x$ on $[1,2]$
4. $f(x) = x^{3/2}$ on $[0,1]$

Answers - Sample Problems

1.) $\dfrac{2}{27}\left(37\sqrt{37}-1\right)$ 2.) 45 3.) $\dfrac{14}{3}$ 4.) $\ln\left(1+\sqrt{2}\right)$

Answers - Practice Problems

1.) $\dfrac{e}{2} - \dfrac{1}{2e}$ 2.) 2π 3.) $3 + \dfrac{1}{8}\ln 2$ 4.) $\dfrac{13\sqrt{13}-8}{27}$

Sample Problems - Solutions

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = 2(x-1)^{3/2}$ on $[1,5]$

   Solution:

$$\begin{aligned} f'(x) &= 2\left(\frac{3}{2}\right)(x-1)^{1/2} = 3\sqrt{x-1} \\ \left[f'(x)\right]^2 &= 9(x-1) = 9x-9 \end{aligned}$$

$$\begin{aligned} L &= \int_1^5 \sqrt{1+[f'(x)]^2}\, dx = \int_1^5 \sqrt{1+9x-9}\, dx = \int_1^5 (9x-8)^{1/2}\, dx = \frac{1}{9}\left(\frac{2}{3}\right)(9x-8)^{3/2}\Big|_1^5 \\ &= \frac{2}{27}\left((9\cdot 5-8)^{3/2} - (9\cdot 1-8)^{3/2}\right) = \frac{2}{27}\left(37^{3/2} - 1^{3/2}\right) = \boxed{\frac{2}{27}\left(37\sqrt{37}-1\right)} \end{aligned}$$

2. $f(x) = \dfrac{2}{3}\left(x^2+1\right)^{3/2}$ on $[1,4]$

   Solution:

$$\begin{aligned} f(x) &= \frac{2}{3}\left(x^2+1\right)^{3/2} \\ f'(x) &= \frac{2}{3}\left(\frac{3}{2}\right)\left(x^2+1\right)^{1/2}(2x) = 2x\sqrt{x^2+1} \\ \left[f'(x)\right]^2+1 &= 4x^2\left(x^2+1\right)+1 = 4x^4+4x^2+1 \\ \sqrt{\left[f'(x)\right]^2+1} &= \sqrt{4x^4+4x^2+1} = \sqrt{\left(2x^2+1\right)^2} = 2x^2+1 \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1+[f'(x)]^2}\, dx = \int_1^4 2x^2+1\, dx = \frac{2x^3}{3}+x\Big|_1^4 = \left(\frac{2\cdot 4^3}{3}+4\right) - \left(\frac{2\cdot 1^3}{3}+1\right) \\ &= \left(\frac{128}{3}+4\right) - \left(\frac{2}{3}+1\right) = \frac{126}{3}+3 = \boxed{45} \end{aligned}$$

3. $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on $[1,3]$

   Solution:

$$\begin{aligned} f(x) &= \frac{x^3}{6} + \frac{1}{2x} \\ f'(x) &= \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) \\ \left[f'(x)\right]^2+1 &= \frac{1}{4}\left(x^4+\frac{1}{x^4}-2\right)+1 = \frac{1}{4}\left(x^4+\frac{1}{x^4}-2\right)+\frac{4}{4} = \frac{1}{4}\left(x^4+\frac{1}{x^4}+2\right) = \left[\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\right]^2 \end{aligned}$$

$$\begin{aligned}
L &= \int_1^3 \sqrt{1+\left[f'(x)\right]^2}\,dx = \int_1^3 \sqrt{\left[\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\right]^2}\,dx = \int_1^3 \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\,dx = \frac{1}{2}\int_1^3 x^2+\frac{1}{x^2}\,dx \\
&= \frac{1}{2}\left(\frac{x^3}{3}-\frac{1}{x}\right)\Big|_1^3 = \frac{1}{2}\left[\left(\frac{3^3}{3}-\frac{1}{3}\right)-\left(\frac{1^3}{3}-\frac{1}{1}\right)\right] = \frac{1}{2}\left[\left(9-\frac{1}{3}\right)-\left(\frac{1}{3}-1\right)\right] \\
&= \frac{1}{2}\left[\frac{26}{3}-\left(-\frac{2}{3}\right)\right] = \frac{1}{2}\left(\frac{28}{3}\right) = \boxed{\frac{14}{3}}
\end{aligned}$$

4. $f(x) = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$

Solution:

$$\begin{aligned}
f(x) &= \ln(\cos x) \\
f'(x) &= \frac{1}{\cos x}(-\sin x) = -\tan x \\
\left[f'(x)\right]^2+1 &= \tan^2 x + 1 = \sec^2 x
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1+\left[f'(x)\right]^2}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \ln\left|\sec x+\tan x\right|\Big|_0^{\pi/4} \\
&= \ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right| - \ln\left|\sec 0+\tan 0\right| = \ln\left|\sqrt{2}+1\right| - \ln\left|1+0\right| == \ln\left(\sqrt{2}+1\right) - \ln 1 \\
&= \boxed{\ln\left(\sqrt{2}+1\right)}
\end{aligned}$$

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