One dimension

1. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 3\,\text{g}$ at $x_1 = 2$ and $m_2 = 7\,\text{g}$ at $x_2 = 6$.

2. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 5\,\text{g}$ at $x_1 = -2$, $m_2 = 8\,\text{g}$ at $x_2 = 1$, and $m_3 = 3\,\text{g}$ at $x_3 = 5$.

3. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 1\,\text{g}$ at $x_1 = 1$, $m_2 = 2\,\text{g}$ at $x_2 = 2$, and $m_3 = 3\,\text{g}$ at $x_3 = 3$.

4. Compute the center of mass of the system consisting of m_1, m_2, ...,m_n located on the number line at x_1, x_2,...,x_n.

5. A rod of length 3 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(0,0)$ and $(3,0)$, then the density of the material at x is given by $\delta(x) = x^2 \dfrac{\text{g}}{\text{cm}}$.
   a) Compute the mass of the rod.
   b) Compute the center of mass of the rod.

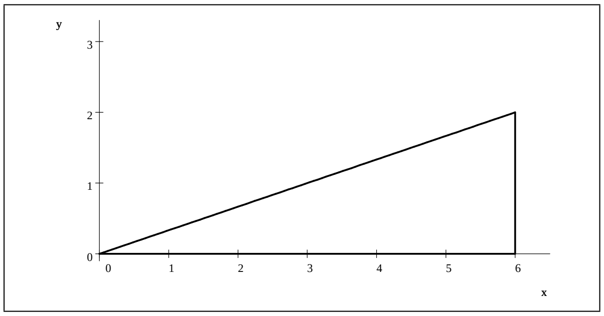
6. A rod of length 5 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(1,0)$ and $(6,0)$, then the density of the material at x is given by $\delta(x) = \dfrac{1}{x^2}\dfrac{\text{g}}{\text{cm}}$.
   a) Compute the mass of the rod.
   b) Compute the center of mass of the rod.

Two dimensions

7. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 13\,\text{g}$ at $P_1(2,-5)$ and $m_2 = 7\,\text{g}$ at $P_2(-3,4)$.

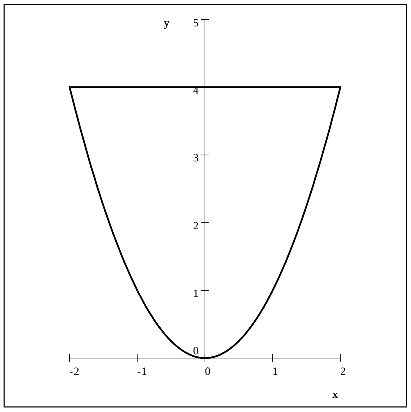
8. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 3\,\text{g}$ at $P_1(-1,1)$, $m_2 = 5\,\text{g}$ at $P_2(4,-3)$, and $m_3 = 7\,\text{g}$ at $P_3(-2,0)$.

9. The triangular plate shown on the picture below is bounded by the graphs of $y = \dfrac{x}{3}$, $y = 0$, and $x = 6$ has a constant density of $5\dfrac{\text{g}}{\text{cm}^2}$.

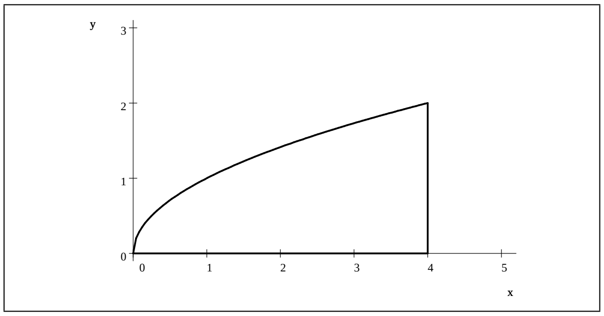


   a) Compute the plate's moment M_x about the y-axis.
   b) Compute the mass of the plate.
   c) Compute the x-coordinate of the center of mass of the plate.
   d) Compute the plate's moment M_y about the x-axis.
   e) Compute the y-coordinate of the center of mass of the plate.

10. Compute the center of mass of a thin plate bounded by the graphs of $y = x^2$ and $y = 4$ between $x = -2$ and $x = 2$.

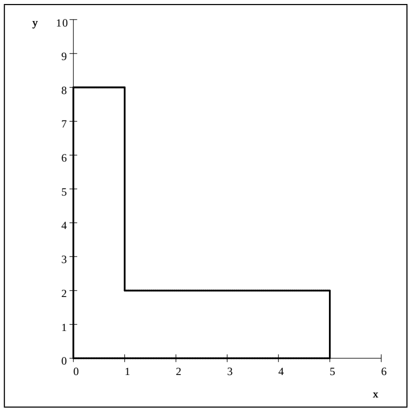


11. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$.



12. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 9$.

13. Compute the center of mass of the object shown on the picture below.



14. Compute the center of mass of the region bounded by the graphs of $y = \dfrac{1}{x^3}$, $x = 1$, and $y = 0$.

Answers

1. 4.8
2. $\frac{13}{16} = 0.812\,5$
3. $\frac{7}{3}$
4. $\dfrac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$
5. a) 9 b) $\frac{9}{4} = 2.25$
6. a) $\frac{5}{6}$ b) $\frac{6}{5}\ln 6$
7. $C\left(\frac{1}{4}, -\frac{37}{20}\right)$
8. $C\left(\frac{1}{5}, -\frac{4}{5}\right)$
9. a) 120 b) 30 c) 4 d) 20 e) $\frac{2}{3}$
10. $\left(0, \frac{12}{5}\right) = (0, 2.4)$
11. $\left(\frac{12}{5}, \frac{3}{4}\right) = (2.4, 0.75)$
12. $\left(\frac{27}{5}, \frac{9}{8}\right) = (5.4, 1.125)$
13. $\left(\frac{7}{4}, \frac{5}{2}\right) = (1.75, 2.5)$
14. $\left(2, \frac{1}{5}\right) = (2, 0.2)$

Solutions - One dimension

1. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 3\,\text{g}$ at $x_1 = 2$ and $m_2 = 7\,\text{g}$ at $x_2 = 6$.

$$\begin{aligned} m_1(x-x_1)+m_2(x-x_2) &= 0 \\ m_1x - m_1x_1 + m_2x - m_2x_2 &= 0 \\ m_1x + m_2x &= m_1x_1 + m_2x_2 \\ x(m_1+m_2) &= m_1x_1 + m_2x_2 \\ x &= \frac{m_1x_1+m_2x_2}{m_1+m_2} = \frac{3\cdot 2 + 7\cdot 6}{3+7} = \frac{48}{10} = 4.8 \end{aligned}$$

2. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 5\,\text{g}$ at $x_1 = -2$, $m_2 = 8\,\text{g}$ at $x_2 = 1$, and $m_3 = 3\,\text{g}$ at $x_3 = 5$.

   Let x be the center of mass.

$$\begin{aligned} m_1(x-x_1)+m_2(x-x_2)+m_3(x-x_3) &= 0 \\ m_1x - m_1x_1 + m_2x - m_2x_2 + m_3x - m_3x_3 &= 0 \\ m_1x + m_2x + m_3x &= m_1x_1 + m_2x_2 + m_3x_3 \\ x(m_1+m_2+m_3) &= m_1x_1 + m_2x_2 + m_3x_3 \\ x &= \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} \end{aligned}$$

$$x = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} = \frac{5(-2)+8\cdot 1+3\cdot 5}{5+8+3} = \frac{13}{16}$$

3. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 1\,\text{g}$ at $x_1 = 1$, $m_2 = 2\,\text{g}$ at $x_2 = 2$, and $m_3 = 3\,\text{g}$ at $x_3 = 3$. $\dfrac{1^2+2^2+3^2}{1+2+3} = \dfrac{7}{3}$

4. Compute the center of mass of the system consisting of m_1, m_2, ...,m_n located on the number line at x_1, x_2,...,x_n. $\dfrac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$

5. A rod of length 3 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(0,0)$ and $(3,0)$, then the density of the material at x is given by $\delta(x) = x^2 \dfrac{\text{g}}{\text{cm}}$.

   a) Compute the mass of the rod. $m = \int_0^3 dm = \int_0^3 \delta(x)\,dx = \int_0^3 x^2 dx = 9$

   b) Compute the center of mass of the rod. $\dfrac{\int_0^3 x\,dm}{\int_0^3 dm} = \dfrac{\int_0^3 x\delta(x)\,dx}{\int_0^3 \delta(x)\,dx} = \dfrac{\int_0^3 x\cdot x^2 dx}{9} = \dfrac{\frac{81}{4}}{9} = \dfrac{9}{4}$

6. A rod of length 5 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(1,0)$ and $(6,0)$, then the density of the material at x is given by $\delta(x) = \dfrac{1}{x^2}\dfrac{\text{g}}{\text{cm}}$.

a) Compute the mass of the rod. $m = \int_1^6 dm = \int_1^6 \delta(x)\,dx = \int_1^6 \frac{1}{x^2}dx = \frac{5}{6}$

b) Compute the center of mass of the rod. $\dfrac{\int_1^6 x\,dm}{\int_1^6 dm} = \dfrac{\int_1^6 x\delta(x)\,dx}{\int_1^6 \delta(x)\,dx} = \dfrac{\int_1^6 x\frac{1}{x^2}dx}{\frac{5}{6}} = \frac{6}{5}\ln 6$

Two dimensions

7. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 13\,\text{g}$ at $P_1(2,-5)$ and $m_2 = 7\,\text{g}$ at $P_2(-3,4)$.

$$\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2} = \frac{13\cdot 2 + 7\cdot(-3)}{13+7} = \frac{1}{4}$$

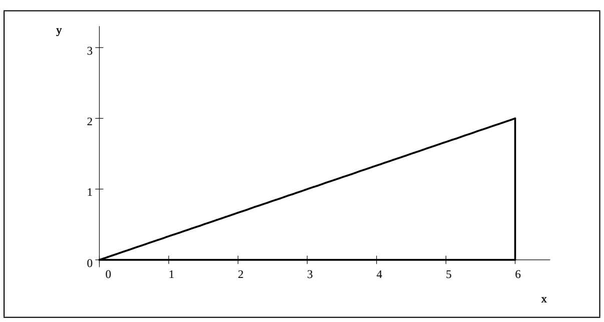
$$\overline{y} = \frac{m_1y_1 + m_2y_2}{m_1+m_2} = \frac{13(-5) + 7\cdot(4)}{13+7} = -\frac{37}{20} \qquad C\left(\frac{1}{4}, -\frac{37}{20}\right)$$

8. Compute the center of mass of the system consisting of m_1, m_2, and m_3 located on the number line if given that $m_1 = 3\,\text{g}$ at $P_1(-1,1)$, $m_2 = 5\,\text{g}$ at $P_2(4,-3)$, and $m_3 = 7\,\text{g}$ at $P_3(-2,0)$.

$$\overline{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1+m_2+m_3} = \frac{3(-1) + 5\cdot 4 + 7(-2)}{3+5+7} = \frac{1}{5}$$

$$\overline{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1+m_2+m_3} = \frac{3\cdot 1 + 5(-3) + 7\cdot 0}{3+5+7} = -\frac{4}{5} \qquad C\left(\frac{1}{5}, -\frac{4}{5}\right)$$

9. The triangular plate shown on the picture below is bounded by the graphs of $y = \dfrac{x}{3}$, $y = 0$, and $x = 6$ has a constant density of $5\dfrac{\text{g}}{\text{cm}^2}$.



a) Compute the plate's moment M_x about the $y-$axis.

Let us first slice the object into very thin vertical slices. In a general slice,

center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \dfrac{x}{6}\right)$

vertical side: $\dfrac{x}{3}$

horizontal side: dx area: $dA = \dfrac{x}{3}dx$

mass: $dm = \delta dA = 5 \cdot \dfrac{x}{3}dx = \dfrac{5x}{3}dx$

The moment of the strip about the $y-$axis: $\widetilde{x}dm = x\left(\dfrac{5x}{3}dx\right) = \dfrac{5}{3}x^2dx$

The moment of the plate about the $y-$axis: $\displaystyle\int \widetilde{x}dm = \int_0^6 \frac{5}{3}x^2dx = 120$

b) Compute the mass of the plate. $\displaystyle M = \int dm = \int_0^6 \frac{5}{3}xdx = 30$

c) Compute the $x-$coordinate of the center of mass of the plate. $\displaystyle \overline{x} = \frac{\int \widetilde{x}dm}{\int dm} = \frac{120}{30} = 4$

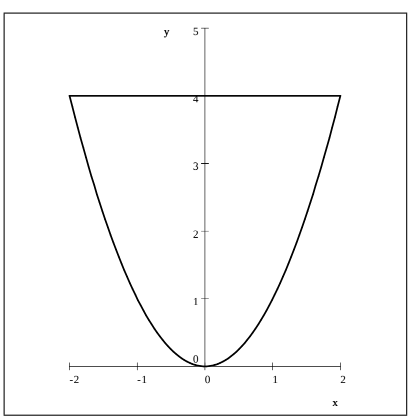
d) Compute the plate's moment M_y about the $x-$axis.

The moment of the strip about the $x-$axis: $\widetilde{y}dm = \dfrac{x}{6}\left(\dfrac{5x}{3}dx\right) = \dfrac{5}{18}x^2dx$

The moment of the plate about the $x-$axis: $\displaystyle\int \widetilde{y}dm = \int_0^6 \frac{5}{18}x^2dx = 20$

e) Compute the $y-$coordinate of the center of mass of the plate. $\displaystyle \overline{y} = \frac{\int \widetilde{y}dm}{\int dm} = \frac{20}{30} = \frac{2}{3}$

10. Compute the center of mass of a thin plate bounded by the graphs of $y = x^2$ and $y = 4$ between $x = -2$ and $x = 2$.



Solution: a general stripe located at x

center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \dfrac{1}{2}\left(x^2+4\right)\right)$

horizontal side: dx

vertical side: $4 - x^2$

area: $dA = \left(4 - x^2\right)dx$

mass: $dm = \delta\left(4 - x^2\right)dx$

total mass of the plate: $\displaystyle\int dm = \int_{-2}^{2} \delta\left(4 - x^2\right)dx = \frac{32}{3}\delta$

moment of stripe about the $y-$axis: $\widetilde{y}dm = x\delta\left(4-x^2\right)dx$

moment of plate about the $y-$axis: $\displaystyle\int \widetilde{x}dm = \int_{-2}^{2} \delta x\left(4-x^2\right)dx = \delta\int_{-2}^{2} x\left(4-x^2\right)dx = 0$

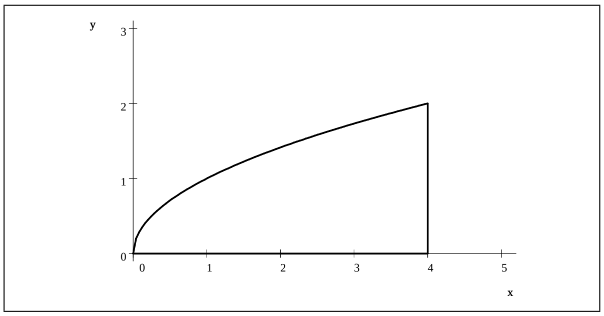
the $x-$coordinate of center of mass: $\displaystyle\overline{x} = \frac{\int \widetilde{x}dm}{\int dm} = \frac{0}{\frac{32}{3}\delta} = 0$

moment of stripe about the $x-$axis: $\widetilde{y}dm = \dfrac{1}{2}\left(x^2+4\right)\delta\left(4-x^2\right)dx$

moment of plate about the $x-$axis: $\displaystyle\int \widetilde{y}dm = \int_{-2}^{2}\frac{\delta}{2}\left(x^2+4\right)\left(4-x^2\right) = \frac{\delta}{2}\int_{-2}^{2}\left(x^2+4\right)\left(4-x^2\right)dx = \frac{128}{5}\delta$

the $y-$coordinate of center of mass: $\displaystyle\overline{y} = \frac{\int \widetilde{y}dm}{\int dm} = \frac{\frac{128}{5}\delta}{\frac{32}{3}\delta} = \frac{12}{5}$

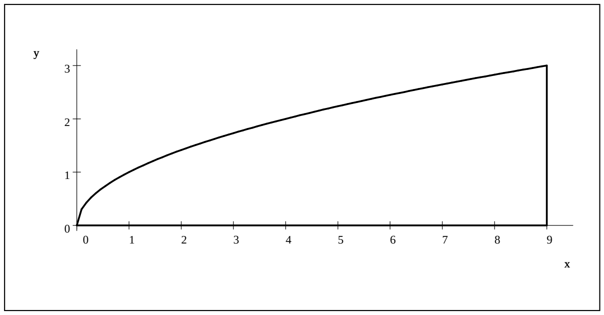
11. Compute the center of mass of the region bounded by the graphs of $y=\sqrt{x}$, $y=0$, and $x=4$.



$x-$coordinate: $\displaystyle\overline{x} = \frac{\int_0^4 x\delta\sqrt{x}dx}{\int_0^4 \delta\sqrt{x}dx} = \frac{\frac{64}{5}\delta}{\frac{16}{3}\delta} = \frac{12}{5} = 2.4$

$y-$coordinate: $\displaystyle\overline{y} = \frac{\int_0^4 \left(\frac{\sqrt{x}}{2}\right)\delta\sqrt{x}dx}{\int_0^4 \delta\sqrt{x}dx} = \frac{4\delta}{\frac{16}{3}\delta} = \frac{3}{4} = 0.75$ $\qquad\left(\dfrac{12}{5},\dfrac{3}{4}\right) = (2.4, 0.75)$

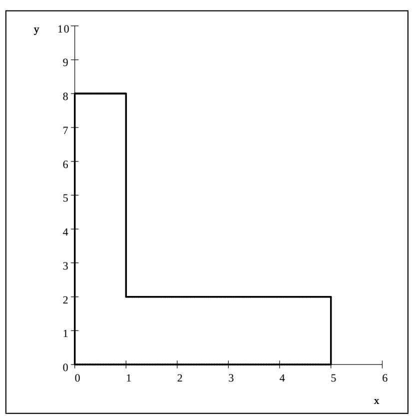
12. Compute the center of mass of the region bounded by the graphs of $y=\sqrt{x}$, $y=0$, and $x=9$.



$x-$coordinate: $$\overline{x}=\frac{\displaystyle\int_0^9 x\delta\sqrt{x}dx}{\displaystyle\int_0^9 \delta\sqrt{x}dx}=\frac{\frac{486}{5}\delta}{18\delta}=\frac{27}{5}=5.4$$

$y-$coordinate: $$\overline{y}=\frac{\displaystyle\int_0^9 \left(\frac{\sqrt{x}}{2}\right)\delta\sqrt{x}dx}{\displaystyle\int_0^9 \delta\sqrt{x}dx}=\frac{\frac{81}{4}\delta}{18\delta}=\frac{9}{8}=1.125 \qquad \left(\frac{27}{5},\frac{9}{8}\right)=(5.4,1.125)$$

13. Compute the center of mass of the object shown on the picture below.



Part 1.

Vertical stripe: dx by 8

mass: $8\delta dx$

mass: $\displaystyle\int_0^1 8\delta dx=8\delta$

moment about the $y-$axis: $\displaystyle\int_0^1 8x\delta dx=4\delta$

moment about the $x-$axis: $\displaystyle\int_0^1 8(4)\delta dx=32\delta$

Part 2:

Vertical stripe: dx by 2

mass: $2\delta dx$

mass: $\displaystyle\int_1^5 2\delta dx = 8\delta$

moment about the $y-$axis: $\displaystyle\int_1^5 2x\delta dx = 24\delta$

moment about the $x-$axis: $\displaystyle\int_1^5 1\,(2)\,\delta dx = 8\delta$

$x-$coordinate of center of mass: $$\frac{\displaystyle\int_0^1 8x\delta dx + \int_1^5 2x\delta dx}{\displaystyle\int_0^1 8\delta dx + \int_1^5 2\delta dx} = \frac{4\delta + 24\delta}{8\delta + 8\delta} = \frac{7}{4}$$

$y-$coordinate of center of mass: $$\frac{\displaystyle\int_0^1 32\delta dx + \int_1^5 2\delta dx}{\displaystyle\int_0^1 8\delta dx + \int_1^5 2\delta dx} = \frac{32\delta + 8\delta}{8\delta + 8\delta} = \frac{5}{2} \qquad C\left(\frac{7}{4}, \frac{5}{2}\right)$$

14. Compute the center of mass of the region bounded by the graphs of $y = \dfrac{1}{x^3}$, $x = 1$, and $y = 0$.

$$\overline{x} = \frac{\displaystyle\int_1^\infty x\frac{1}{x^3}dx}{\displaystyle\int_1^\infty \frac{1}{x^3}dx} = 2 \qquad \overline{y} = \frac{\displaystyle\int_1^\infty \left(\frac{1}{2x^3}\right)\frac{1}{x^3}dx}{\displaystyle\int_1^\infty \frac{1}{x^3}dx} = \frac{1}{5} \qquad \left(2, \frac{1}{5}\right)$$