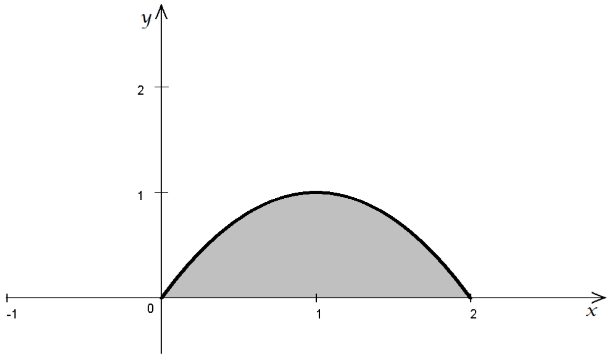
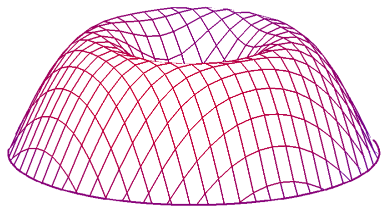
# Sample Problems

1. Consider the region bounded by $f(x) = -x^2 + 2x$, $y = 0$, and $x = 2$. Compute the volume of the object we obtain when rotating this region about the $y-$axis.



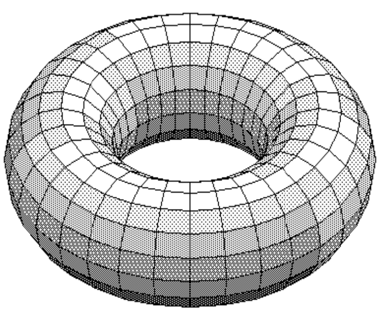



2. Compute the volume of the object that we obtain by rotating the region between $y = \frac{1}{3}x$, $y = 0$, and $x = 6$

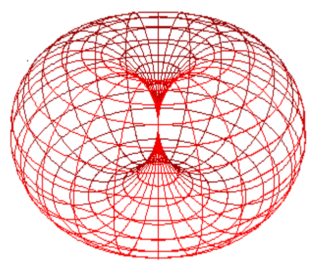
   a) about the $y-$axis  b) about the line $x = -1$

3. Compute the volume of the object that we obtain by rotating the region between $y = \sin x$, $y = 0$, and $x = \pi$ about the $y-$axis.

4. (the torus) Consider the region bounded by the circle $(x-2)^2 + y^2 = 1$. Compute the volume of the object we obtain when rotating the region



$a)$



$b)$

   a) about the $y-$axis

   b) about the line $x = 1$

   c) Assume that $R > r$. A circle is centered at $(R, 0)$ and has radius $r$. We rotate it about the $y-$axis. Compute the volume of this torus.

# Practice Problems

1. Compute the volume of the object we obtain by rotating the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the $y-$axis.
2. Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the $y-$axis.
3. Compute the volume of the object we obtain by rotating the region bounded by $y = \dfrac{1}{x}$, $y = 0$, $x = 1$ and $x = 2$, about the $y-$axis.
4. a) The torus created by rotating the unit circle centered at $(1, 0)$ about the $y-$axis.

   b) The torus created by rotating the unit circle centered at $(2, 0)$ about the $y-$axis.

# Answers - Sample Problems

1.) $\dfrac{8}{3}\pi$   2.) a) $48\pi$   b) $60\pi$   3.) $2\pi^2$   4.) a) $4\pi^2$   b) $2\pi^2$   c) $V = 2\pi^2 r^2 R$

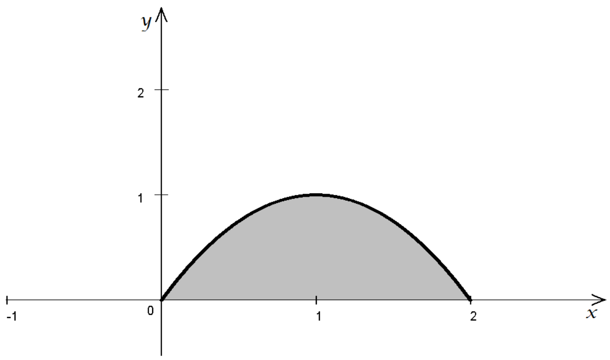
# Answers - Practice Problems

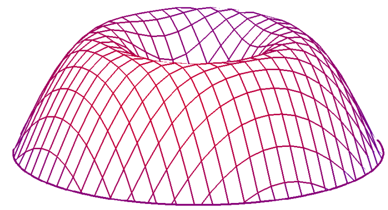
1.) $\dfrac{\pi}{2}$   2.) $\dfrac{128}{5}\pi$   3.) $2\pi$   4.) a) $2\pi^2$   b) $4\pi^2$

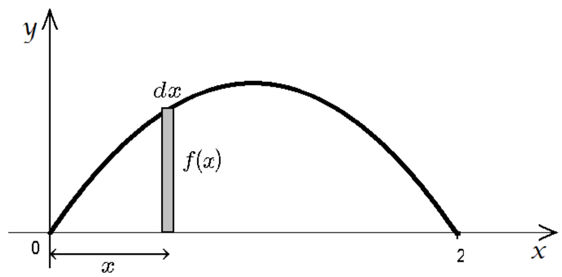
# Solutions of Sample Problems

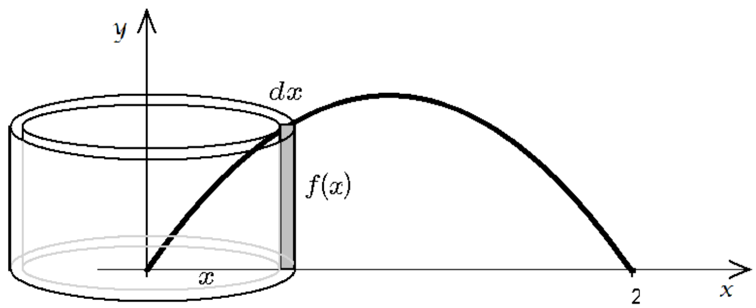
1. Consider the region bounded by $f(x) = -x^2 + 2x$, $y = 0$, and $x = 2$. Compute the volume of the object we obtain when rotating this region about the $y-$axis.



Solution: We partition the interval $[0, 2]$ into very small intervals over which we approximate the function to be constant. The area under the graph becomes a sum of many rectangles. Consider rotating one such rectangle about the $y-$axis. The object we obtain is called a cylindrical shell.




The volume of such a cylindrical shell is approximated by

$$V_i = \text{circumference} \cdot \text{height} \cdot \text{thickness}$$

In this particular case, the circumference is $2\pi x$, the height is $f(x)$ , and the thickness is $dx$. The total volume is then

$$\begin{aligned} V &= \int_0^2 (2\pi x) f(x)\, dx = \int_0^2 2\pi x\left(-x^2+2x\right) dx = 2\pi \int_0^2 -x^3 + 2x^2\, dx = 2\pi\left(\frac{-x^4}{4} + \frac{2x^3}{3}\right)\Bigg|_0^2 \\ &= 2\pi\left(\frac{-16}{4} + \frac{16}{3}\right) = 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32}{12}\pi = \boxed{\frac{8}{3}\pi} \end{aligned}$$

2. Compute the volume of the object that we obtain by rotating the region between $y = \frac{1}{3}x$, $y = 0$, and $x = 6$

   a) about the $y-$axis

   Solution:

$$V = \int_0^6 (2\pi x) f(x)\, dx = \int_0^6 2\pi x\left(\frac{1}{3}x\right) dx = \frac{2\pi}{3}\int_0^6 x^2\, dx = \frac{2\pi}{3}\left(\frac{x^3}{3}\right)\Bigg|_0^6 = \frac{2\pi}{9}\left(6^3 - 0\right) = \boxed{48\pi}$$

b) about the line $x = -1$

Solution: If we rotate around the line $x = -1$, then the cylindrical shell at $x$ will have a radius of $x + 1$.

$$V = \int_0^6 2\pi(x+1) f(x)\ dx = \int_0^6 2\pi(x+1)\left(\frac{1}{3}x\right)\ dx = \frac{2\pi}{3}\int_0^6 x^2 + x\ dx = \frac{2\pi}{3}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_0^6$$
$$= \frac{2\pi}{3}\left[\left(\frac{6^3}{3} + \frac{6^2}{2}\right) - 0\right] = \frac{2\pi}{3}(90) = \boxed{60\pi}$$

3. Compute the volume of the object that we obtain by rotating the region between $y = \sin x$, $y = 0$, and $x = \pi$ about the $y-$axis.

Solution:
$$V = \int_0^\pi 2\pi(x) f(x)\ dx = \int_0^\pi 2\pi x(\sin x)\ dx = 2\pi\int_0^\pi x\ \sin x\ dx$$

We will first evaluatre the indefinite integral by integrating by parts. We will use the formula $\int f'g = fg - \int fg'$.

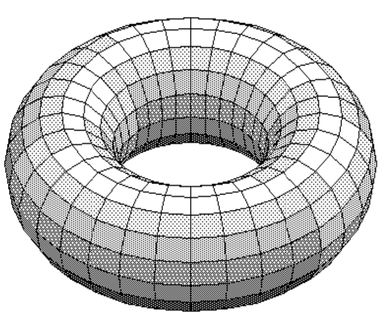
Let $g(x) = x$ and $f'(x) = \sin x$. Then also $g'(x) = 1$ and $f(x) = -\cos x$ and so

$$\int f'g = fg - \int fg' \text{ becomes}$$
$$\int x \sin x\ dx = x(-\cos x) - \int -\cos x du = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

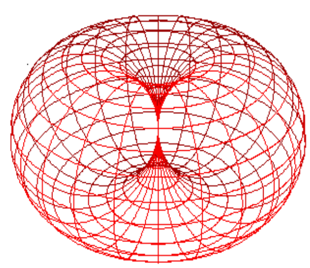
We are now ready to compute the definite integral

$$V = 2\pi\int_0^\pi x\ \sin x\ dx = 2\pi(-x\cos x + \sin x)\Big|_0^\pi = 2\pi[(-\pi\cos\pi + \sin\pi) - (-0\cos 0 + \sin 0)]$$
$$= 2\pi[(-\pi(-1) + 0) - (0)] = \boxed{2\pi^2}$$

4. (the torus) Consider the region bounded by the circle $(x-2)^2 + y^2 = 1$. Compute the volume of the object we obtain when rotating the region



$a)$



$b)$

a) about the $y-$axis

Solution: We solve for $y$ in the equation and obtain $y = \pm\sqrt{1-(x-2)^2}$. We will rotate the region between the $x-$axis and $f(x) = \sqrt{1-(x-2)^2}$ (which is the upper half) and then multiply the volume by 2.

$$V = 2\int_1^3 2\pi x f(x)\ dx = 4\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$$

Let us first substitute $u = x-2$. Then clearly $x = u+2$ and $dx = du$. We will need to substitute the limits of the integral as well. When $x = 1$, then $u = 1-2 = -1$ and when $x = 3$, then $u = 3-2 = 1$. Now the volume is

$$\begin{aligned} V &= 4\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx = 4\pi\int_{-1}^1 (u+2)\sqrt{1-u^2}\,du = 4\pi\int_{-1}^1 u\sqrt{1-u^2}+2\sqrt{1-u^2}\,du \\ &= 4\pi\left(\int_{-1}^1 u\sqrt{1-u^2}\,du + \int_{-1}^1 2\sqrt{1-u^2}\,du\right) \end{aligned}$$

The two integrals can be computed via different methods. The first integral, $\int_{-1}^1 u\sqrt{1-u^2}\,du$ is zero because the integrand is an odd function and the limits are opposites. (Computation will yield for the same result, just substitute $m = u^2$.) The second integral, $\int_{-1}^1 2\sqrt{1-u^2}\,du = 2\int_{-1}^1 \sqrt{1-u^2}\,du$ is $\pi(1)^2 = \pi$ because it expresses the area of the unit circle. (Recall that if we replace $f(x)$ by $y$ in $f(x) = \sqrt{1-x^2}$ and square both siedes, we get $x^2+y^2 = 1$.) The computation for $\int_{-1}^1 \sqrt{1-u^2}\,du = \frac{\pi}{2}$ can be found in the handout Integrating by Substitution. Thus the volume is

$$V = 4\pi\left(\int_{-1}^1 u\sqrt{1-u^2}\,du + \int_{-1}^1 2\sqrt{1-u^2}\,du\right) = 4\pi(0+\pi) = \boxed{4\pi^2}$$

b) about the line $x = 1$

Solution:

$$V = 2\int_1^3 (2\pi)(x-1) f(x)\ dx = 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$

Again we will substitute $u = x-2$. As before, $x = u+2$ and $dx = du$. We will need to substitute the limits of the integral as well. When $x = 1$, then $u = 1-2 = -1$ and when $x = 3$, then $u = 3-2 = 1$. Now the volume is

$$\begin{aligned} V &= 4\pi\int_1^3 (u+2-1)\sqrt{1-u^2}\,du = 4\pi\int_{-1}^1 (u+1)\sqrt{1-u^2}\,du = 4\pi\left(\int_{-1}^1 u\sqrt{1-u^2}+\sqrt{1-u^2}\,du\right) \\ &= 4\pi\left(\int_{-1}^1 u\sqrt{1-u^2}\,du + \int_{-1}^1 \sqrt{1-u^2}\,du\right) \end{aligned}$$

The first integral is zero because the integrand is odd and the limits are opposites. The second integral $\int_{-1}^1 \sqrt{1-u^2}\,du$ is

$\frac{\pi}{2}$ because $\int\limits_{-1}^{1} \sqrt{1-u^2}$ is the area under a semicircle of radius 1, which is $\frac{\pi}{2}$. (The computation for $\int\limits_{-1}^{1} \sqrt{1-u^2}\,du = \frac{\pi}{2}$ can be found in the handout Integrating by Substitution.) Putting all these together,

$$V = 4\pi\left(\int\limits_{-1}^{1} u\sqrt{1-u^2}\,du + \int\limits_{-1}^{1} \sqrt{1-u^2}\,du\right) = 4\pi\left(0+\frac{\pi}{2}\right) = \boxed{2\pi^2}$$

c) Assume that $R > r$. A circle is centered at $(R,0)$ and has radius $r$. We rotate it about the $y-$axis. Compute the volume of this torus.

Solution: We will use shells to compute the volume we obtain when we rotate the upper semicircle. We get the right volume if we just double that.

$$V = 2\int\limits_{R-r}^{R+r} 2\pi x\sqrt{r^2-(x-R)^2}\,dx = 4\pi\int\limits_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\,dx$$

Let $u = x - R$. Then of course $du = dx$ and $x = u + R$. We will substitute the limits as well.

$$\begin{aligned} V &= 4\pi\int\limits_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\,dx = 4\pi\int\limits_{-r}^{r} (u+R)\sqrt{r^2-u^2}\,du = 4\pi\int\limits_{-r}^{r} u\sqrt{r^2-u^2} + R\sqrt{r^2-u^2}\,du \\ &= 4\pi\left(\int\limits_{-r}^{r} u\sqrt{r^2-u^2}\,du + \int\limits_{-r}^{r} R\sqrt{r^2-u^2}\,du\right) \end{aligned}$$

The two integrals are substantially different. The first one is zero since the function $u\sqrt{r^2-u^2}$ is odd and the limits are opposites. The second integral is

$$\int\limits_{-r}^{r} R\sqrt{r^2-u^2}\,du = R\int\limits_{-r}^{r} \sqrt{r^2-u^2}\,du = R\left(\frac{1}{2}\pi r^2\right)$$

because the integral now expresses the area under a semicircle with radius $r$. (The computation for $\int\limits_{-r}^{r} \sqrt{r^2-u^2}\,du = \frac{1}{2}\pi r^2$ can be found in the handout Integrating by Substitution.) The volume is

$$V = 4\pi\left(\int\limits_{-r}^{r} u\sqrt{r^2-u^2}\,du + \int\limits_{-r}^{r} R\sqrt{r^2-u^2}\,du\right) = 4\pi\left(0+\frac{1}{2}\pi r^2 R\right) = \boxed{2\pi^2 r^2 R}$$

Notice that we rotated a circle of radius $r$ in a circular motion with radius $R$. It is interesting that according to this formula, we can think of the result as multiplying the area of the small circle by the perimeter of the big one: $(\pi r^2)(2\pi R) = 2\pi^2 r^2 R$.