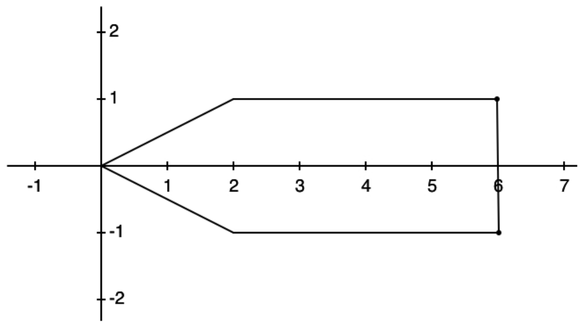
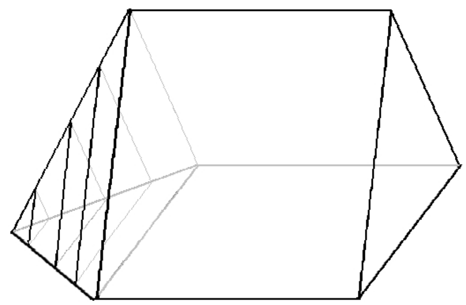
Computing Volumes by Adding Cross Sections

Sample Problems

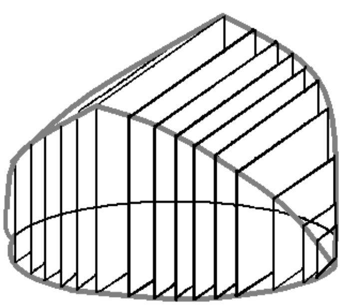
1. Find the volume of the pyramid with a square base if its base has sides 10 meters long and its height is 12 meters.

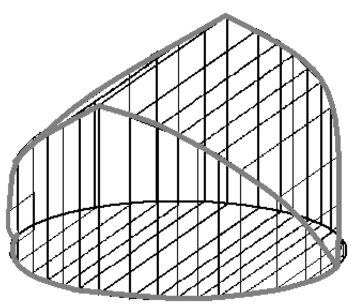
2. A building's base is a rectangle and a triangle as shown on the picture below. At each point, the vertical cross section of the building is an equilateral triangle. Compute the volume of the building.



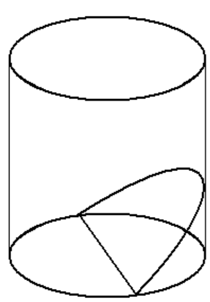


3. Find the volume of the solid whose base is a circle with radius 1 and each cross-section perpendicular to the base is a square.





4. A wedge is cut out of a circular cylinder of radius 5 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Compute the volume of the wedge.



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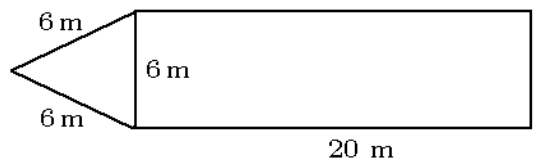
Practice Problems

1. a) Find the volume of the pyramid with a square base if its base has sides 20 meters long and its height is 15 meters.
   b) We want to cut the pyramid by a plane parallel to its base. At what height should we cut if we wanted to cut the pyramid into two parts of equal volume?

2. a) Use integration to compute the volume of a pyramid with a square base of sides L long and has a height of H.
   b) We want to cut the pyramid by a plane parallel to its base. At what height should we cut if we wanted to cut the pyramid into two parts of equal volume?

3. Compute the volume of the building if its base is shown on the picture below and

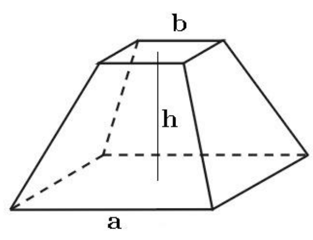


   a) cross-sections perpendicular to the base are squares.
   b) cross-sections perpendicular to the base are equilateral triangles.
   c) cross-sections perpendicular to the base are semi-circles.
   d) cross-sections perpendicular to the base are isosceles right triangles where the hypotenuse lies on the base.

4. Consider a solid with a circular base of radius 1. Cross-sections perpendicular to the base are equilateral triangle. Compute the volume of the solid.

5. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of $60°$ along a diameter of the cylinder. Compute the volume of the wedge.

6. Use integration to compute the volume of the frustum of a pyramid shown on the picture below.



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Sample Problems - Answers

1.) 400 2.) $\dfrac{14\sqrt{3}}{3}$ 3.) $\dfrac{16}{3}$ 4.) $\dfrac{250\sqrt{3}}{9}$

Practice Problems - Answers

1.) a) $2000\,\mathrm{m}^3$ b) $15-\sqrt[3]{\dfrac{3375}{2}} \approx 3.0945\,\mathrm{m}$ from the ground

2.) a) $\dfrac{1}{3}HL^2$ b) $\left(1-\sqrt[3]{\dfrac{1}{2}}\right)H$ from the ground

3.) a) $720+36\sqrt{3}$ b) $27+180\sqrt{3}$ c) $\left(90+\dfrac{9}{2}\sqrt{3}\right)\pi$ d) $180+9\sqrt{3}$

4.) $\dfrac{4\sqrt{3}}{3}$ 5.) $\dfrac{128\sqrt{3}}{3}$ 6.) $V=\dfrac{1}{3}h\left(a^2+ab+b^2\right)$

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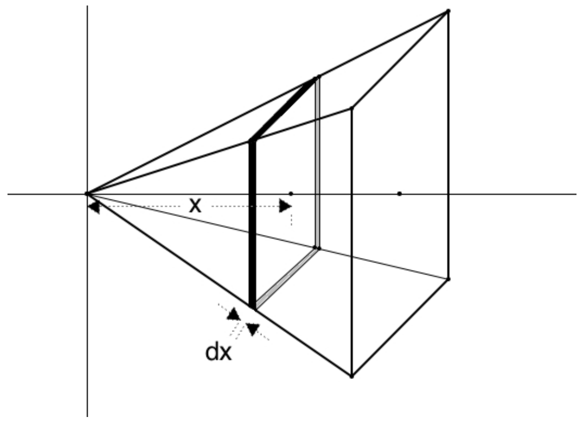
Sample Problems - Solutions

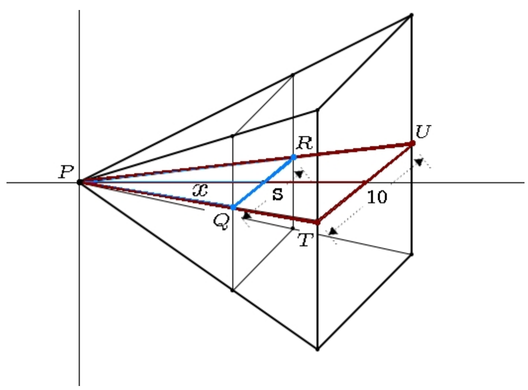
1. Find the volume of the pyramid with a square base if its base has sides 10 meters long and its height is 12 meters.
Solution:



Consider the picture above. We will slice the pyramid into thin slices perpendicular to the x−axis. Each slice has volume $A(x)\,dx$ where $A(x)$ is the area of the cross section. We will add the volume of these slices ranging from $x = 0$ to $x = 12$. The volume is

$$V = \int_0^{12} A(x)\,dx$$

We now need to figure out $A(x)$. The area of a square is s^2 where s is the length of its side.



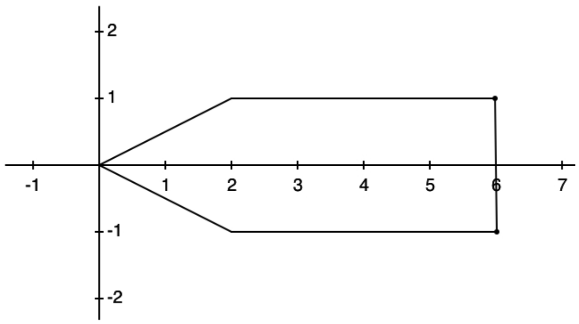
Consider the similar triangles PQR and PTU. Their heights are x and 12, correspondingly. Their bases are s and 10, correspondingly. The ratio between these four quantites will give us s as a function of x.

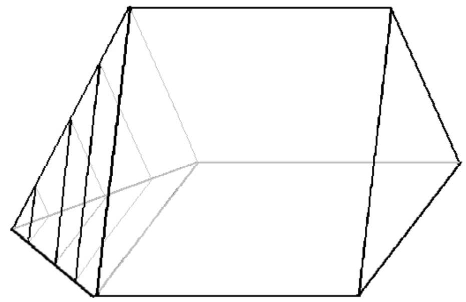
$$\frac{s}{x} = \frac{10}{12} \qquad \Longrightarrow \qquad s = \frac{5}{6}x \quad \text{and so} \quad A(x) = \left(\frac{5}{6}x\right)^2 = \frac{25}{36}x^2$$

Thus

$$V = \int_0^{12} A(x)\,dx = \int_0^{12} \frac{25}{36}x^2 dx = \frac{25}{36}\int_0^{12} x^2 dx = \frac{25}{36}\,\frac{x^3}{3}\bigg|_0^{12} = \frac{25}{36}\left(\frac{12^3}{3} - \frac{0^3}{3}\right) = \frac{25}{36}(576) = \boxed{400}$$

2. A building's base is a rectangle and a triangle as shown on the picture below. At each point, the vertical cross section of the building is an equilateral triangle. Compute the volume of the building.

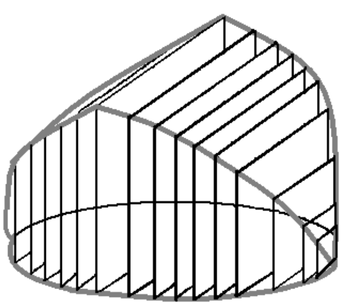


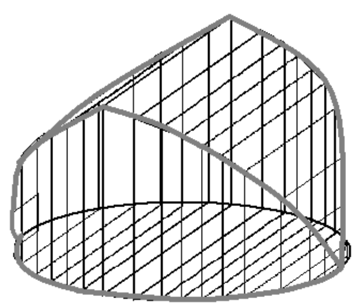


Solution: If an equilateral triangle's side is s, then its area is $A = \frac{1}{2}s\left(\frac{\sqrt{3}}{2}s\right) = \frac{\sqrt{3}}{4}s^2$. Between 0 and 2, the side of the triangle at x is x. Between 2 and 6, each triangle has sides 2.

$$\begin{aligned} V &= \int_0^6 A(x)\,dx = \int_0^6 \frac{\sqrt{3}}{4}[s(x)]^2\,dx = \int_0^2 \frac{\sqrt{3}}{4}x^2\,dx + \int_2^6 \frac{\sqrt{3}}{4}\cdot 2^2\,dx = \frac{\sqrt{3}}{4}\int_0^2 x^2\,dx + \sqrt{3}\int_2^6 1\,dx \\ &= \frac{\sqrt{3}}{4}\frac{x^3}{3}\bigg|_0^2 + \sqrt{3}x\bigg|_2^6 = \frac{\sqrt{3}}{12}\left(2^3 - 0^3\right) + \sqrt{3}\,(6-2) = \sqrt{3}\left(\frac{8}{12}+4\right) = \boxed{\frac{14\sqrt{3}}{3}} \end{aligned}$$

3. Find the volume of the solid whose base is a circle with radius 1 and each cross-section perpendicular to the base is a square.





Solution: Let us assume that the circle is centered at the origin; that is, its equation is $x^2+y^2=1$. For each value of x, the cross section is a square with sides $s = 2y = 2\sqrt{1-x^2}$. Tus the area of each cross section is

$$A(x) = s^2 = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right)$$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} 4\left(1-x^2\right)dx = 4\int_{-1}^{1}\left(1-x^2\right)dx = 8\int_0^1\left(1-x^2\right)dx = 8\left(x-\frac{x^3}{3}\right)\bigg|_0^1 = 8\left(1-\frac{1^3}{3}\right) = \boxed{\frac{16}{3}}$$

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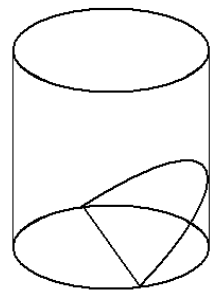
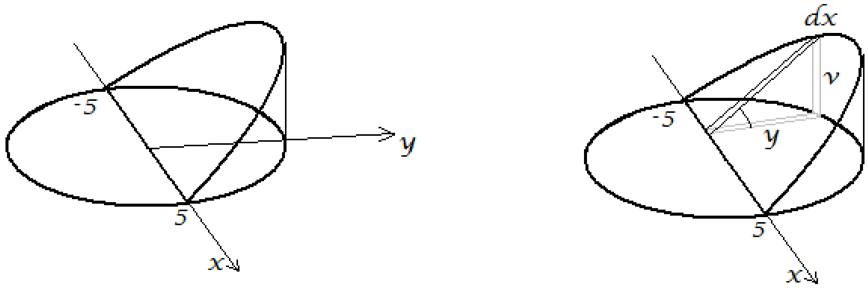
4. A wedge is cut out of a circular cylinder of radius 5 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Compute the volume of the wedge.



Solution: We place the coordinate system so that the $x-$axis is the intersection of the two planes and the $y-$axis also lies in the base of the wedge.



We slice the wedge into very thin slices along the $x-$axis. The slices range from $x = -5$ to $x = 5$ but we already see that the wedge is symmetrical to the $y-$axis, so we can just add the volume of the slices from $x = 0$ to $x = 5$ and double our result.

Each cross section has thickness dx and is in the shape of a right triangle with the right angle lying on the circle. The angle lying on the $x-$axis is 30°. The horizontal side is y where $y = \sqrt{25 - x^2}$. The vertical side, denoted by v is

$$\tan 30^\circ = \frac{v}{y} \quad \Longrightarrow \quad v = y \tan 30^\circ = \frac{\sqrt{3}}{3} y = \frac{\sqrt{3}}{3}\sqrt{25 - x^2}$$

The volume of a triangular slice is

$$dV = A dx = \frac{1}{2} y v dx = \frac{1}{2}\left(\underbrace{\sqrt{25 - x^2}}_{y}\right)\left(\underbrace{\frac{\sqrt{3}}{3}\sqrt{25 - x^2}}_{v}\right) dx = \frac{\sqrt{3}}{6}\left(25 - x^2\right) dx$$

The total volume is then

$$\begin{aligned} V &= \int dV = \int_{-5}^{5} \frac{\sqrt{3}}{6}\left(25 - x^2\right) dx = 2\int_{0}^{5} \frac{\sqrt{3}}{6}\left(25 - x^2\right) dx = 2 \cdot \frac{\sqrt{3}}{6}\int_{0}^{5} 25 - x^2 \, dx = \frac{\sqrt{3}}{3}\left(25x - \frac{x^3}{3}\bigg|_0^5\right) \\ &= \frac{\sqrt{3}}{3}\left(\left(25 \cdot 5 - \frac{5^3}{3}\right) - \left(25 \cdot 0 - \frac{0^3}{3}\right)\right) = \frac{\sqrt{3}}{3}\left(125 - \frac{125}{3}\right) = \frac{\sqrt{3}}{3} \cdot 125\left(1 - \frac{1}{3}\right) \\ &= \frac{\sqrt{3}}{3} \cdot 125 \cdot \frac{2}{3} = \boxed{\frac{250\sqrt{3}}{9}} \end{aligned}$$

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