Sample Problems

1. Consider the solid obtained by revolving the region under the graph of $y = \sqrt{x}$ on $[0, 4]$ about the $x-$axis.

   a) Let us use the partition $\{x_0, x_1, ..., x_n\}$ and approximate the function as constant within each subinterval. Let us rotate this approximation about the $x-$axis. Write an expression for V_i, the volume of the ith small disk.

   b) Write a Riemann sum approximating the volume of the solid.

   c) Find an integral expressing the volume and evaluate the integral.

2. Consider the solid obtained by revolving the region under the graph of $y = \frac{1}{2}x$ on $[0, 10]$ about the $x-$axis.

   a) Write a Riemann sum expressing the volume.

   b) Find an integral expressing the volume and evaluate the integral.

3. Consider the solid obtained by revolving the region under the graph of $y = mx$ on $[0, h]$ about the $x-$axis.

   a) Write a Riemann sum expressing the volume.

   b) Find an integral expressing the volume and evaluate the integral.

   c) This object is a cone with height h and base radius $r = mh$. Express the volume computed in part b) in terms of r and h.

4. Compute the volume of the solid obtained by revolving the region under the graph of $y = \sqrt{R^2 - x^2}$ on $[-R, R]$ about the $x-$axis.

5. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the $y-$axis.

6. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$ about the $y-$axis.

7. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \arctan x$, $y = \frac{\pi}{4}$, and $x = 0$ about the $y-$axis.

8. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$ about the $x-$axis.

9. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $x = 0$, and $y = 4$ about the $y-$axis.

Practice Problems

1. Find the volume of the solid obtained by rotating the region bounded by $y = \sin x$, $y = 0$ between $x = 0$ and $x = \pi$ about the $x-$axis.

2. Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\ln x}$, $y = 0$, $x = 1$, and $x = e$ about the $x-$axis.

3. Compute the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$ about the $x-$axis.

4. Compute the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $x = 1$, and $y = 0$ about the $x-$axis. (This will be an improper integral.)

5. Compute the volume of the solid obtained by rotating the region bounded by $y = \operatorname{sech} x$, $y = 0$, $x = 0$ and $x = 1$ about the $x-$axis.

6. a) Compute the volume of the solid obtained by rotating the region bounded by $y = \dfrac{x}{3}$, $y = 0$, and $x = 6$ about the $x-$axis.

   b) Compute the volume of the solid obtained by rotating the region bounded by $y = \dfrac{x}{3}$, $y = 2$, and $x = 0$ about the $y-$axis.

7. a) Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt[4]{x}$, $y = 0$, and $x = 16$ about the $x-$axis.

   b) Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$ about the $y-$axis.

8. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

Sample Problems - Answers

1.) a) $V_i = \pi \left[f(x_i)\right]^2 \Delta x_i$ b) $\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \pi \left[f(x_i)\right]^2 \Delta x_i$ c) 8π

2.) a) $\sum_{i=1}^{n} \pi \left[f(x_i)\right]^2 \Delta x_i$ b) $\frac{250}{3}\pi$ 3.) a) $\sum_{i=1}^{n} \pi \left[f(x_i)\right]^2 \Delta x_i$ b) $\frac{1}{3}\pi m^2 h^3$ c) $\frac{1}{3}\pi r^2 h$

4.) $\frac{4}{3}\pi R^3$ 5.) $\frac{32}{5}\pi$ 6.) $\frac{96}{5}\pi$ 7.) $\pi - \frac{\pi^2}{4}$ 8.) $\frac{32}{5}\pi$ 9.) 8π

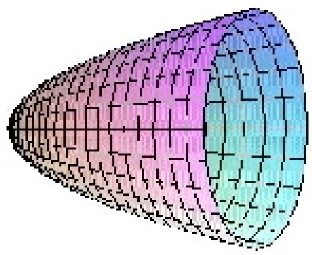
Practice Problems - Answers

1.) $\frac{\pi^2}{2}$ 2.) π 3.) $\frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$ 4.) π 5.) $\frac{e^2-1}{e^2+1}\pi$ 6.) a) 8π b) 24π

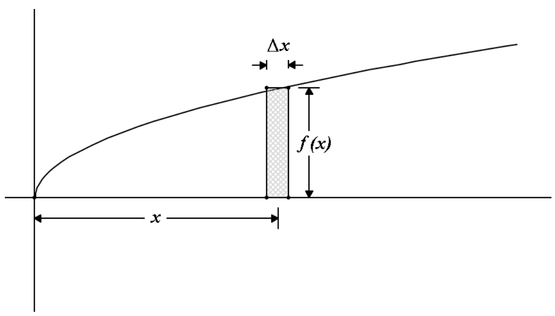
7.) a) $\frac{128}{3}\pi$ b) $\frac{512}{9}\pi$ 8.) $\frac{5}{12}\pi r^3$

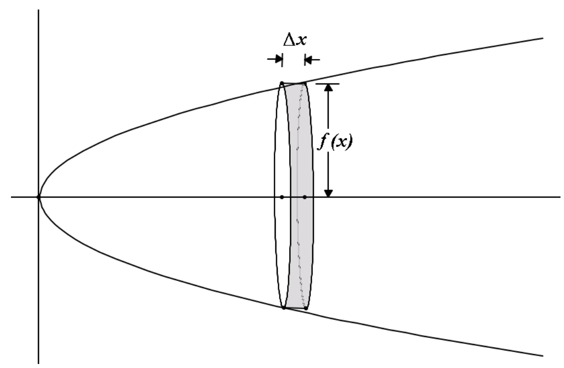
Sample Problems - Solutions

1. Consider the solid obtained by revolving the region under the graph of $y = \sqrt{x}$ on $[0,4]$ about the $x-$axis.



a) Let us use the partition $\{x_0, x_1, ..., x_n\}$ and approximate the function as constant within each subinterval. Let us rotate this approximation about the $x-$axis. Write an expression for V_i, the volume of the ith small disk.

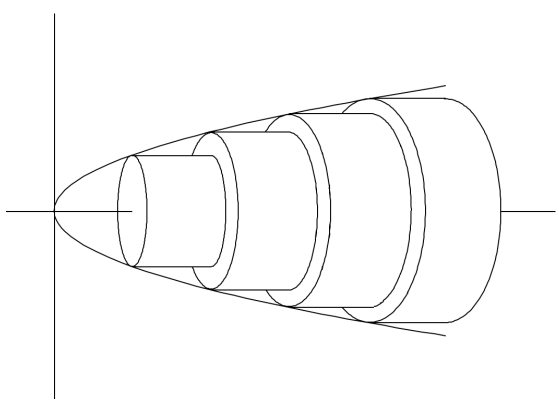


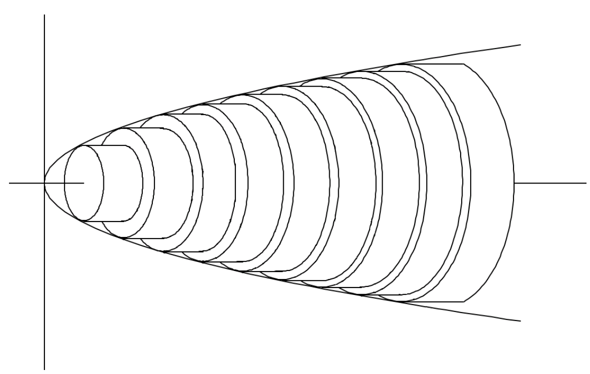


Solution: One thin disk is created by rotating the rectangle shown on the picture above. The rectangle has height $f(x)$ and width Δx. When we rotate this rectangle, we obtain a cylinder with radius $f(x)$ and height Δx. A cylinder's volume is $V = \pi r^2 h$. In this case,

$$V_i = \pi [f(x_i)]^2 \Delta x_i$$

b) Write a Riemann sum approximating the volume of the solid.





$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \pi [f(x_i)]^2 \Delta x_i$$

c) Find an integral expressing the volume and evaluate the integral.
Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[f(x_i)\right]^2\Delta x_i = \int_0^4 \pi\left[f(x)\right]^2\ dx = \pi\int_0^4\left(\sqrt{x}\right)^2\ dx = \pi\int_0^4 x\ dx = \pi\left.\frac{x^2}{2}\right|_0^4 = \frac{\pi}{2}\left(4^2-0^2\right) \\ &= \frac{\pi}{2}(16) = 8\pi \end{aligned}$$

2. Consider the solid obtained by revolving the region under the graph of $y=\dfrac{1}{2}x$ on $[0,10]$ about the $x-$axis.

a) Write a Riemann sum expressing the volume.
Solution: Let x be fixed. One thin disk is created by rotating the rectangle shown on the picture above. A cylinder's volume is $V=\pi r^2 h$. In this case, $\Delta V=\pi\left[f(x_i)\right]^2\Delta x_i$. The Riemann sum is then

$$\sum_{i=1}^{n}\pi\left[f(x_i)\right]^2\Delta x_i$$

b) Find an integral expressing the volume and evaluate the integral.
Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[f(x_i)\right]^2\Delta x_i = \int_0^{10}\pi\left[f(x)\right]^2\ dx = \int_0^{10}\pi\left(\frac{x}{2}\right)^2\ dx = \int_0^{10}\frac{\pi}{4}x^2\ dx = \frac{\pi}{4}\int_0^{10}x^2\ dx \\ &= \frac{\pi}{4}\left(\left.\frac{x^3}{3}\right|_0^{10}\right) = \frac{\pi}{12}\left(10^3-0^3\right) = \frac{\pi}{12}(1000) = \frac{250}{3}\pi \end{aligned}$$

3. Consider the solid obtained by revolving the region under the graph of $y=mx$ on $[0,h]$ about the $x-$axis.

a) Write a Riemann sum expressing the volume.
Solution: Let x be fixed. The ith thin disk is created by rotating a rectangle with width Δx_i and height $f(x_i)$. The volume of such a cylinder is $V_i=\pi\left[f(x_i)\right]^2\Delta x_i$. The Riemann sum is then

$$\sum_{i=1}^{n}\pi\left[f(x_i)\right]^2\Delta x_i$$

b) Find an integral expressing the volume and evaluate the integral.
Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[f(x_i)\right]^2\Delta x_i = \int_0^{h}\pi\left[f(x)\right]^2\Delta x = \int_0^{h}\pi m^2x^2\ dx = \pi m^2\int_0^{h}x^2\ dx = \pi m^2\left(\left.\frac{x^3}{3}\right|_0^{h}\right) \\ &= \frac{\pi m^2}{3}\left(h^3-0^3\right) = \frac{1}{3}\pi m^2h^3 \end{aligned}$$

c) This object is a cone with height h and base radius $r=mh$. Express the volume computed in part b) in terms of r and h.

Solution:
$$V=\frac{1}{3}\pi m^2h^3=\frac{1}{3}\pi\left(m^2h^2\right)h=\frac{1}{3}\pi(mh)^2h=\frac{1}{3}\pi r^2h$$

4. Compute the volume of the solid obtained by revolving the region under the graph of $y = \sqrt{R^2 - x^2}$ on $[-R, R]$ about the $x-$axis.

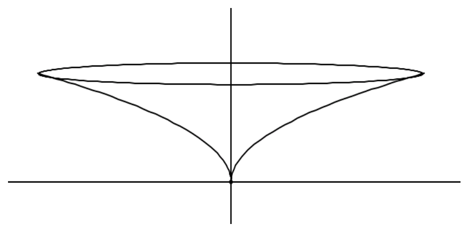
Solution:

$$V_i = \pi \left[f(x_i)\right]^2 \Delta x_i \qquad \text{Riemann sum} = \sum_{i=1}^{n} \pi \left[f(x_i)\right]^2 \Delta x_i$$

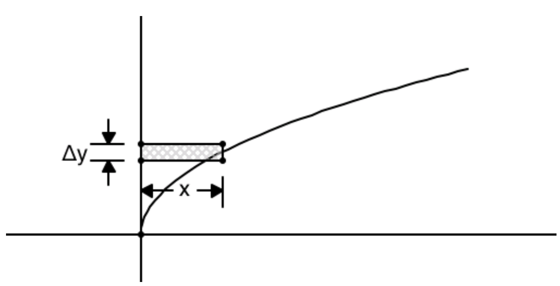
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[f(x_i)\right]^2 \Delta x_i = \int_{-R}^{R} \pi \left[f(x)\right]^2 dx = \pi \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx = \pi \int_{-R}^{R} R^2 - x^2 \, dx$$

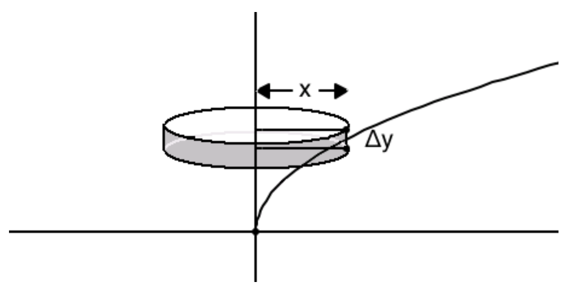
$$= 2\pi \int_{0}^{R} R^2 - x^2 \, dx = 2\pi \left(R^2 x - \frac{x^3}{3}\right)\Bigg|_0^R = 2\pi \left(R^3 - \frac{R^3}{3}\right) = 2\pi \left(\frac{2}{3} R^3\right) = \frac{4}{3}\pi R^3$$

5. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the $y-$axis.



Solution: We will again rotate rectangles to obtain cylinders but this time we rotate them about the $y-$axis.





We will need to express everything in terms of y. If $y = \sqrt{x}$, then $x = y^2$.

The volume of one disk is $V_i = \pi x_i^2 \Delta y_i$. then the Riemann sum is $\sum_{i=1}^{n} \pi x_i^2 \Delta y_i = \sum_{i=1}^{n} \pi \left[f^{-1}(y_i)\right]^2 \Delta y_i$. The volume is the limit of the Riemann sum as n approaches infinity.

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi x_i^2 \Delta y_i = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[f^{-1}(y_i)\right]^2 \Delta y_i = \int_0^2 \pi \left[f^{-1}(y)\right]^2 dy = \int_0^2 \pi \left(y^2\right)^2 dy = \pi \int_0^2 y^4 dy$$

$$= \pi \frac{y^5}{5}\Bigg|_0^2 = \frac{\pi}{5}\left(2^5 - 0^5\right) = \frac{32}{5}\pi$$

6. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$ about the $y-$axis.
Solution: If $y = x^3$, then $x = \sqrt[3]{y}$. We will need to express everything in terms of y.
The volume of one disk is $V_i = \pi x_i^2 \Delta y_i$. Then the entire object's volume is

$$\begin{aligned} V &= \int_0^8 \pi x^2 dy = \int_0^8 \pi \left(\sqrt[3]{y}\right)^2 dy = \pi \int_0^8 y^{2/3} dy = \pi \left(\frac{3}{5}\right) y^{5/3} \Big|_0^8 = \frac{3}{5}\pi \left(8^{5/3} - 0^{5/3}\right) \\ &= \frac{3}{5}\pi (32) = \frac{96}{5}\pi \end{aligned}$$

7. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \arctan x$, $y = \dfrac{\pi}{4}$, and $x = 0$ about the $y-$axis.
Solution: If $y = \arctan x$, then $x = \tan y$. We will need to express everything in terms of y.
The volume of one disk is $V_i = \pi x^2 dy$. Then the entire object's volume is

$$V = \int_0^{\pi/4} \pi x^2 dy = \int_0^{\pi/4} \pi (\tan y)^2 dy = \pi \int_0^{\pi/4} \tan^2 y dy$$

Recall that

$$\int \tan^2 y dy = \int \sec^2 y - 1 \ dy = \tan y - y + C$$

$$\begin{aligned} \pi \int_0^{\pi/4} \tan^2 y \ dy &= \pi (\tan y - y) \Big|_0^{\pi/4} = \pi \left[\left(\tan \frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right] = \pi \left[\left(1 - \frac{\pi}{4}\right) - 0\right] \\ &= \pi \left(1 - \frac{\pi}{4}\right) = \pi - \frac{\pi^2}{4} \end{aligned}$$

8. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$ about the $x-$axis.
Solution:

$$\int_0^2 \pi [f(x)]^2 \ dx = \int_0^2 \pi \left(x^2\right)^2 \ dx = \pi \int_0^2 x^4 \ dx = \pi \left. \frac{x^5}{5} \right|_0^2 = \frac{\pi}{5} \left(2^5 - 0^5\right) = \frac{32}{5}\pi$$

9. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $x = 0$, and $y = 4$ about the $y-$axis.
Solution: $x = \sqrt{y}$

$$\int_0^4 \pi (x)^2 \ dy = \int_0^4 \pi \left(\sqrt{y}\right)^2 \ dy = \pi \int_0^4 y \ dy = \pi \left. \frac{y^2}{2} \right|_0^4 = \frac{\pi}{2} \left(4^2 - 0^2\right) = 8\pi$$

