Sample Problems

1. a) Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ about the $x-$axis.

   b) Compute the volume of the object we obtain if we revolve the region about the $y-$axis.

2. The region bounded by $y = x^2 + 1$ and $y = x + 3$ is rotated about the $x-$axis. Find the volume of this object.

3. Let R be the region bounded by $y = x$, $y = 2x$, and $y = 6$. Find the volume of the object we obtain by revolving R about

   a) the $y-$axis b) the $x-$axis

Practice Problems

1. Let R be the region bounded by $y = x$ and $y = 2\sqrt{x}$. Find the volume of the object we obtain by revolving R about

   a) the $x-$axis b) the $y-$axis

2. Let R be the region in the first quadrant bounded by $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the $x-$axis or the $y-$axis?

   Let R be the region bounded by $y = x$, $y = x + 2$, $x = 0$, and $x = 4$. Find the volume of the object we obtain by revolving R about

   a) the $x-$axis b*) the $y-$axis

3. The region bounded by $y = \cos x$ and $y = \sin x$ (between $x = 0$ and $x = \dfrac{\pi}{4}$) is rotated about the $x-$axis. Find the volume of this object.

4. Let R be the region bounded by $y = 0$, $y = \sqrt{\ln x}$, $y = 2$, and $x = 0$. Find the volume of the object we obtain by revolving R about the $x-$axis.

Answers - Sample Problems

1.) a) $\frac{8}{3}\pi$ b) $\frac{64}{15}\pi$ 2.) $\frac{117}{5}\pi$ 3.) a) 54π b) 72π

Answers - Practice Problems

1.) a) $\frac{32}{3}\pi$ b) $\frac{128}{15}\pi$

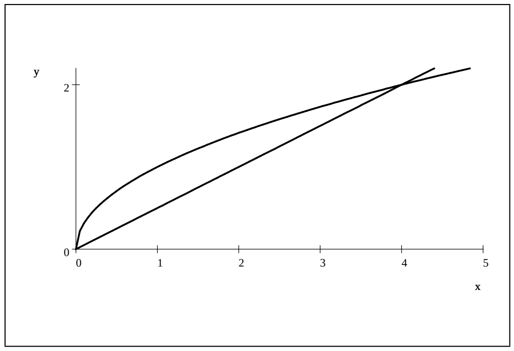
2.) about the $x-$axis: $V = \frac{128}{15}\pi \approx 26.808\,257$ about the $y-$axis: $V = \frac{512}{21}\pi \approx 76.595\,021$

3.) a) 48π b*) 32π 4.) $\frac{\pi}{2}$ 5.) $\pi\left(e^4 - 1\right)$

Sample Problems - Solutions

1. a) Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ about the $x-$axis.

Solution: We first solve for the intersection of the two graphs and obtain the intersection points $(0,0)$ and $(4,2)$. We are now ready to sketch the two graphs together.



The volume can be computed as

$$\begin{aligned} V &= \int_0^4 \pi\left(\left(\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2\right)dx = \pi\int_0^4 \left(x - \frac{x^2}{4}\right)dx = \pi\left(\left(\frac{x^2}{2} - \frac{x^3}{12}\right)\Big|_0^4\right) \\ &= \pi\left(\left(\frac{4^2}{2} - \frac{4^3}{12}\right) - \left(\frac{0^2}{2} - \frac{0^3}{12}\right)\right) = \pi\left(\left(8 - \frac{64}{12}\right) - 0\right) = \pi\left(8 - \frac{16}{3}\right) = \boxed{\frac{8}{3}\pi} \end{aligned}$$

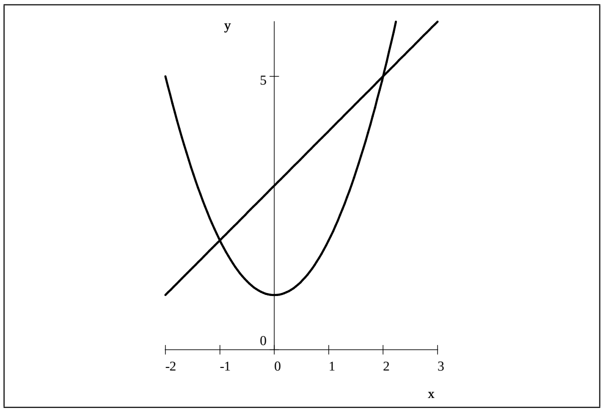
b) Compute the volume of the object we obtain if we revolve the region about the $y-$axis.

Solution: Now the volume can be computed as $\int \pi\left(x_1^2 - x_2^2\right)dy$ where we express x_1 and x_2 in terms of y. The limits of the integration are the range where the horizontal slices are located. In this case, between $y = 0$ and $y = 2$. We will also need to solve $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ for x. Clearly, if $y = \sqrt{x}$, then $x = y^2$ and if $y = \dfrac{x}{2}$, then $x = 2y$. Also notice that now the graph of $y = \dfrac{x}{2}$ is farther from the center of rotation. So the integral for the volume is

$$\begin{aligned} V &= \int_0^2 \pi\left(x_1^2 - x_2^2\right)dy = \pi\int_0^2 (2y)^2 - \left(y^2\right)^2 dy = \pi\int_0^2 4y^2 - y^4 dy = \pi\left(\frac{4y^3}{3} - \frac{y^5}{5}\right)\Big|_0^2 \\ &= \pi\left(\left(\frac{4\cdot 2^3}{3} - \frac{2^5}{5}\right) - \left(\frac{4\cdot 0^3}{3} - \frac{0^5}{5}\right)\right) = \pi\left(\left(\frac{32}{3} - \frac{32}{5}\right) - 0\right) = 32\pi\left(\frac{1}{3} - \frac{1}{5}\right) \\ &= 32\pi\left(\frac{2}{15}\right) = \boxed{\frac{64}{15}\pi} \end{aligned}$$

2. The region bounded by $y = x^2 + 1$ and $y = x + 3$ is rotated about the $x-$axis. Find the volume of this object.

   Solution: We first solve for the intersection points and obtain the points $(-1, 2)$ and $(2, 5)$.



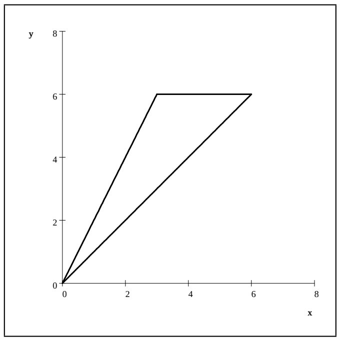
The volume can be computed as

$$\begin{aligned}
V &= \int \pi\left(y_2^2 - y_1^2\right)dx = \int_{-1}^{2} \pi\left((x+3)^2 - \left(x^2+1\right)^2\right)dx = \pi\int_{-1}^{2}\left(x^2+6x+9-\left(x^4+2x^2+1\right)\right)dx \\
&= \pi\int_{-1}^{2}\left(x^2+6x+9-x^4-2x^2-1\right)dx = \pi\int_{-1}^{2}\left(-x^4-x^2+6x+8\right)dx = \pi\left(-\frac{x^5}{5}-\frac{x^3}{3}+3x^2+8x\right)\Bigg|_{-1}^{2} \\
&= \pi\left(\left(-\frac{2^5}{5}-\frac{2^3}{3}+3\cdot 2^2+8\cdot 2\right)-\left(-\frac{(-1)^5}{5}-\frac{(-1)^3}{3}+3(-1)^2+8(-1)\right)\right) \\
&= \pi\left(\left(-\frac{32}{5}-\frac{8}{3}+12+16\right)-\left(\frac{1}{5}+\frac{1}{3}+3-8\right)\right) = \pi\left(\left(-\frac{136}{15}+28\right)-\left(\frac{8}{15}-5\right)\right) \\
&= \pi\left(-\frac{136}{15}+28-\frac{8}{15}+5\right) = \pi\left(-\frac{144}{15}+33\right) = \pi\left(33-\frac{48}{5}\right) = \boxed{\frac{117}{5}\pi}
\end{aligned}$$

3. Let R be the region bounded by $y = x$, $y = 2x$, and $y = 6$. Find the volume of the object we obtain by revolving R about

   a) the $y-$axis

   Solution:

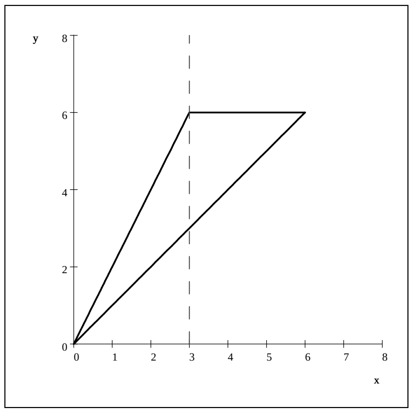


After we graphed the region, we see that $y = x$ is farther from the center of rotation. We also need to express x in terms of y for the integral. If $y = 2x$, then clearly $x = \frac{y}{2}$, and so the volume is

$$\begin{aligned} V &= \int \pi \left(x_2^2 - x_1^2\right) dy = \int_0^6 \pi \left(y^2 - \left(\frac{y}{2}\right)^2\right) dy = \pi \int_0^6 y^2 - \frac{y^2}{4} dy = \pi \int_0^6 \frac{3}{4} y^2 dy = \frac{\pi}{4} \int_0^6 3y^2 dy \\ &= \frac{\pi}{4} \left(y^3 \Big|_0^6 \right) = \frac{\pi}{4} \left(6^3 - 0^3\right) = \frac{216}{4} \pi = \boxed{54\pi} \end{aligned}$$

b) the $x-$axis

Solution: In this case, the washer method works but we need to break down the region into two parts, as shown on the picture below. Between 0 and 3, the graph of y_2 is $y = 2x$. But between 3 and 6, the graph of y_2 is $y = 6$.



The volume is then

$$\begin{aligned} V &= \int \pi \left(y_2^2 - y_1^2\right) dx = \int_0^3 \pi \left((2x)^2 - x^2\right) dx + \int_3^6 \pi \left(6^2 - x^2\right) dx = \pi \int_0^3 \left(4x^2 - x^2\right) dx + \pi \int_3^6 -x^2 + 36 dx \\ &= \pi \int_0^3 3x^2 dx + \pi \int_3^6 -x^2 + 36 dx = \pi \left(\left(x^3 \Big|_0^3 \right) + \left(-\frac{x^3}{3} + 36x \Big|_3^6 \right) \right) \\ &= \pi \left(\left(3^3 - 0^3\right) + \left(-\frac{6^3}{3} + 36 \cdot 6\right) - \left(-\frac{3^3}{3} + 36 \cdot 3\right) \right) = \pi \left(27 + (-72 + 216) - (-9 + 108)\right) \\ &= \pi \left(27 + 144 - 99\right) = \boxed{72\pi} \end{aligned}$$

