Sample Problems

1. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2\frac{\text{N}}{\text{m}}$.

   a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.

   b) Turn the Riemann sum into an integral and evaluate it to find the work.

2. A cylindrical tank has height 10 m and base radius 3 m and it is standing on its circular base. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that $1\,\text{m}^3$ of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000\frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

   a) Find a Riemann sum expressing the work.

   b) Turn the Riemann sum into an integral and evaluate it.

3. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that $1\,\text{m}^3$ of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000\frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

   a) Find a Riemann sum expressing the work.

   b) Turn the Riemann sum into an integral and evaluate it.

4. The gravitational force between two objects can be computed as $F_{gr} = \frac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67 \times 10^{-11}\frac{\text{N}\,\text{m}^2}{\text{kg}^2}$. We want to lift a satellite, measuring 800 kg, from the surface of the Earth to a height of 5000 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of 5.97×10^{24} kg.)

Practice Problems

1. We are standing on the top of a 100 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 500 N and the rope's unit weight is $1.5\frac{\text{N}}{\text{m}}$. Compute the work required to lift the bucket from the ground to the top of the building.

2. A cylindrical tank is of height 8 m and base radius 3 m, and it is standing on its circular base. It is full of oil, and we want to pump it all out by a pipe that is always leveled at the surface of the oil. Assume that $1\,\text{m}^3$ of oil weighs 9 000 N, i.e. the density of oil is $\delta = 9\,000\frac{\text{N}}{\text{m}^3}$.

   a) How much work does it require to pump all oil out of the tank?

   b) How much work does it require to pump all oil out of the tank if the tank is only half full?

3. A cylindrical tank is of height H and base radius R, and it is standing on its circular base. It contains some liquid of density δ, the water level at h. We want to pump out the liquid by a pipe that is always leveled at the surface of the liquid. Compute the formula for the work required to pump all liquid out of the tank.

4. A tank, shaped like a cone has height 12 m and base radius 5 m. It is placed so that the circular part is downward. It is full of water, and we want to pump out by a pipe that is always leveled at the surface of the water. Assume that the density of water is $\delta = 10\,000\dfrac{\mathrm{N}}{\mathrm{m}^3}$.
   a) How much work does it require to pump all water out of the tank?
   b) How much work does it require to pump all water out of the tank if it contains water to half of its height?
5. A tank, shaped like a cone has height H and base radius R. It is placed so that the circular part is downward. It is full of water, and we want to pump out by a pipe that is always leveled at the surface of the water. Assume that the density of water is δ. Compute the formula for the work required to pump all liquid out of the tank.
6. The gravitational force between two objects can be computed as $F_{gr} = \dfrac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67 \times 10^{-11}\dfrac{\mathrm{N\,m}^2}{\mathrm{kg}^2}$. We want to lift a satellite, measuring 500 kg, from the surface of the Earth to a height of 20 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of 5.97×10^{24} kg.)
7. A spherical tank of radius R is full of liquid of density δ. We want to pump out the liquid by a pipe that is always leveled at the surface of the liquid. Compute the formula for the work required to pump all liquid out of the tank.

Answers - Sample Problems

1. a) $W = \sum_{i=0}^{n-1} (370 - 2x_i)\,\Delta x$ b) $\int_0^{60} (370 - 2x)\,dx = 18\,600\ \mathrm{J}$
2. a) $W = \sum_{i=0}^{n-1} \left(\delta\pi r^2\right)(10 - x_i)\,\Delta x$ b) $\int_0^{10} \left(\delta\pi r^2\right)(10 - x)\,dx = 4500\,000\pi\ \mathrm{J} \approx 1.413\,72 \times 10^7\ \mathrm{J}$
3. a) $W = \sum_{i=0}^{n-1} \dfrac{9\delta\pi x_i^2}{100}(10 - x_i)\,\Delta x$ b) $\int_0^{10} \dfrac{9\delta\pi x^2}{100}(10 - x)\,dx = 750\,000\pi\ \mathrm{J} \approx 2.356\,2 \times 10^6\ \mathrm{J}$
4. $2.199\,2 \times 10^{10}$ J

Answers - Practice Problems

1.) 57 500 J 2.) a) $2592\,000\pi\ \mathrm{J} \approx 8.143 \times 10^6\ \mathrm{J}$ b) $1944\,000\pi\ \mathrm{J} \approx 6.107\,3 \times 10^6\ \mathrm{J}$

3.) $\delta r^2 h\pi\left(H - \dfrac{h}{2}\right)$ 4.) a) $9000\,000\pi\ \mathrm{J} \approx 2.827\,4 \times 10^7\ \mathrm{J}$ b) $8437\,500\pi\ \mathrm{J} \approx 2.650\,7 \times 10^7\ \mathrm{J}$

5.) $\dfrac{1}{4}\pi H^2 R^2 \delta$ 6.) $9.782\,7 \times 10^7$ J 7.) $\dfrac{4}{3}\pi R^4 \delta$

Sample Problems - Solutions

1. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2\dfrac{\text{N}}{\text{m}}$.

a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.

Solution: Let x represent the height of the bucket. Let us partition the interval $[0, 60]$ into n equal intervals by $0 = x_0,\ x_1,\ x_2, ..., x_n = 60$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the height of the bucket to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert while lifting the bucket from height x_i to x_{i+1} is

$$F_i = F_{\text{bucket}} + F_{\text{rope}} = 250 + 2\,(60 - x_i) = 370 - 2x_i$$

Then the work we have to invest when lifting the bucket from height x_i to x_{i+1} is

$$W_i = F_i \cdot s_i = (370 - 2x_i)\,(x_{i+1} - x_i) = (370 - 2x_i)\,\Delta x$$

The total work is then

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (370 - 2x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it to find the work.
Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=0}^{n-1} (370 - 2x_i)\,\Delta x = \int_0^{60} (370 - 2x)\,dx = 370x - x^2 \Big|_0^{60} \\ &= \left(370\,(60) - (60)^2\right) - \left(370\,(0) - (0)^2\right) = 22\,200 - 3600 = 18\,600 \end{aligned}$$

or, the same computation with the units, quite elegant,

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(370\,\text{N} - 2\frac{\text{N}}{\text{m}} x_i\right) \Delta x = \int_0^{60\,\text{m}} \left(370\,\text{N} - 2\frac{\text{N}}{\text{m}} x\right) dx = \left(370\,\text{N}x - x^2\,\frac{\text{N}}{\text{m}}\right)\Bigg|_{0\,\text{m}}^{60\,\text{m}} \\ &= \left(370\,\text{N}\,(60\,\text{m}) - (60\,\text{m})^2\,\frac{\text{N}}{\text{m}}\right) - \left(370\,\text{N}\,(0\,\text{m}) - (0\,\text{m})^2\,\frac{\text{N}}{\text{m}}\right) = 22\,200\,\text{N m} - 3600\,\text{N m} = \\ &= 18\,600\,\text{N m} = 18\,600\,\text{J} \end{aligned}$$

2. A cylindrical tank has height 10 m and base radius 3 m and it is standing on its circular base. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that $1\,\mathrm{m}^3$ of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \dfrac{\mathrm{N}}{\mathrm{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0,\ x_1,\ x_{2,} ..., x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume

$$V_i = \pi r^2 \Delta x \quad \text{the weight is then} \quad F_i = \delta V_i = \delta \pi r^2 \Delta x$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = \left(\delta \pi r^2 \Delta x\right)(10 - x_i) = \left(\delta \pi r^2\right)(10 - x_i)\,\Delta x$$

The total work is then is

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \left(\delta \pi r^2\right)(10 - x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

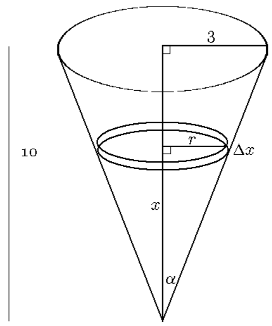
Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned}
W &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\delta \pi r^2\right)(10 - x_i)\,\Delta x = \int_0^{10} \left(\delta \pi r^2\right)(10 - x)\,dx = \delta \pi r^2 \int_0^{10} (10 - x)\,dx \\
&= 10\,000\pi\,(9) \int_0^{10} (10 - x)\,dx = 90\,000\pi \left.\left(10x - \frac{x^2}{2}\right)\right|_0^{10} \\
&= 90\,000\pi \left[\left(10\,(10) - \frac{(10)^2}{2}\right) - \left(10\,(0) - \frac{(0)^2}{2}\right)\right] = 90\,000\pi\ \ (50) \\
&= 4500\,000\pi \approx 1.413\,72 \times 10^7\ \mathrm{J}
\end{aligned}$$

3. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that $1\,\mathrm{m}^3$ of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \dfrac{\mathrm{N}}{\mathrm{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0,\ x_1,\ x_{2,\ldots},x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$.



We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume $\pi r_i^2 h$ where $h = \Delta x$ and r_i is the radius at height x_i. Clearly $\tan\alpha = \dfrac{r_i}{x_i} = \dfrac{3}{10}$ which gives us that $r_i = \dfrac{3}{10} x_i$

$$V_i = \pi r_i^2 \Delta x = \pi \left(\frac{3}{10} x_i\right)^2 \Delta x = \frac{9\pi x_i^2}{100} \Delta x \text{ the weight is then } F_i = \delta V_i = \frac{9\delta\pi x_i^2}{100} \Delta x$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = \left(\frac{9\delta\pi x_i^2}{100} \Delta x\right)(10 - x_i) = \frac{9\delta\pi x_i^2}{100}\ (10 - x_i)\,\Delta x$$

The total work is then is

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100}\ (10 - x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

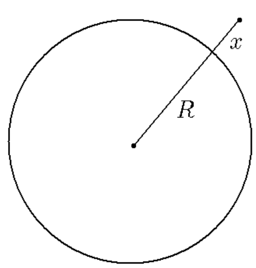
$$\begin{aligned} W &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{9\delta\pi x_i^2}{100}(10-x_i)\,\Delta x = \int_0^{10}\frac{9\delta\pi x^2}{100}(10-x)\,dx = \frac{9\delta\pi}{100}\int_0^{10}x^2(10-x)\,dx \\ &= \frac{9\,(10\,000)\,\pi}{100}\int_0^{10}x^2(10-x)\,dx = 900\pi\int_0^{10}\left(10x^2-x^3\right)dx = 900\pi\left(\frac{10x^3}{3}-\frac{x^4}{4}\right)\Big|_0^{10} \\ &= 900\pi\left[\left(\frac{10\,(10)^3}{3}-\frac{10^4}{4}\right)-\left(\frac{0^3}{3}-\frac{0^4}{4}\right)\right] = 900\pi\left(\frac{10^4}{3}-\frac{10^4}{4}\right) = 900\pi\left(\frac{1}{12}\right)10^4 \\ &= 750\,000\pi \end{aligned}$$

4. The gravitational force between two objects can be computed as $F_{gr} = \dfrac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67\times10^{-11}\dfrac{\mathrm{N\,m^2}}{\mathrm{kg}^2}$. We wanted to lift a satellite, measuring 800 kg, from the surface of the Earth to a height of 5000 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of 5.97×10^{24} kg.)

Solution: Let us denote the radius of Earth by R, the mass of the sattelite by m and that of the Earth by M. Suppose that the satellite is at a distance x from the surface, and we compute the work it requires to move the satellite just a little bit further, by a distance of dx.

$$dW = F_{gr}\cdot dx$$

At a distance of x from the surface, the distance between the centers of masses is $R+x$.



The gravitational force between the satellite and Earth is then

$$F_{gr} = \frac{mMG}{(x+R)^2}$$

and so the work required to lift the satellite by dx is

$$dW = F_{gr}\cdot dx = \frac{mMG}{(x+R)^2}dx$$

The total work required to lift the satellite from the surface to a height of H is then

$$\begin{aligned} W &= \int_0^H \frac{mMG}{(x+R)^2}dx = mMG\int_0^H \frac{1}{(x+R)^2}dx = mMG\left(-\frac{1}{x+R}\bigg|_0^H\right) = mMG\left(-\frac{1}{H+R}-\left(-\frac{1}{0+R}\right)\right) \\ &= mMG\left(\frac{1}{R}-\frac{1}{H+R}\right) = 800\,\mathrm{kg}\cdot 5.97\times 10^{24}\,\mathrm{kg}\cdot 6.67\times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\left(\frac{1}{6370\,\mathrm{m}}-\frac{1}{6370\,\mathrm{m}+5000\,\mathrm{m}}\right) \\ &= 3.1856\times 10^{17}\,\mathrm{N\,m^2}\left(\frac{1}{6370\,\mathrm{km}}-\frac{1}{11370\,\mathrm{km}}\right) = 3.1856\times 10^{17}\,\mathrm{N\,m^2}\left(\frac{1}{6370\,000\,\mathrm{m}}-\frac{1}{11\,370\,000\,\mathrm{m}}\right) \\ &\approx 2.1992\times 10^{10}\,\mathrm{N\,m} \end{aligned}$$

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