Sample Problems

Compute each of the following improper integrals.
Here is a fact you will need for problem 10.): $\frac{d}{dx}(x\ln x - x) = \ln x$.

1. $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{10}^\infty \frac{1}{x\ln x}\,dx$
4. $\displaystyle\int_0^\infty e^{-5x}\,dx$
5. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$
6. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$
7. $\displaystyle\int_0^1 \frac{1}{x}dx$
8. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$
9. $\displaystyle\int_0^1 \ln x\,dx$
10. $\displaystyle\int_{-1}^4 \frac{x}{x^2-9}dx$
11. $\displaystyle\int_0^{25} \frac{1}{\sqrt{x}}dx$
12. $\displaystyle\int_1^2 \frac{1}{\sqrt{2-x}}dx$
13. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{2-x}}dx$

Practice Problems

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
3. $\displaystyle\int_2^\infty e^{-5x}\,dx$
4. $\displaystyle\int_0^1 \frac{1}{x}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
6. $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$
7. $\displaystyle\int_5^\infty \frac{1}{x\ln x}\,dx$
8. $\displaystyle\int_0^1 \frac{1}{x\ln x}\,dx$
9. $\displaystyle\int_0^\infty xe^{-2x^2}\,dx$
10. $\displaystyle\int_1^2 \frac{x^2}{x^3-8}\,dx$
11. $\displaystyle\int_1^\infty \frac{1}{x^2+3}\,dx$
12. $\displaystyle\int_0^{\pi/2} \tan x\,dx$
13. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$
14. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

Answers - Sample Problems

1.) $\frac{1}{3}$ 2.) ∞ 3.) ∞ 4.) $\frac{1}{5}$ 5.) $\frac{1}{2}$ 6.) π 7.) ∞ 8.) 2 9.) -1

10.) undefined 11.) 10 12.) 2 13.) $-\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$

Answers - Practice Problems

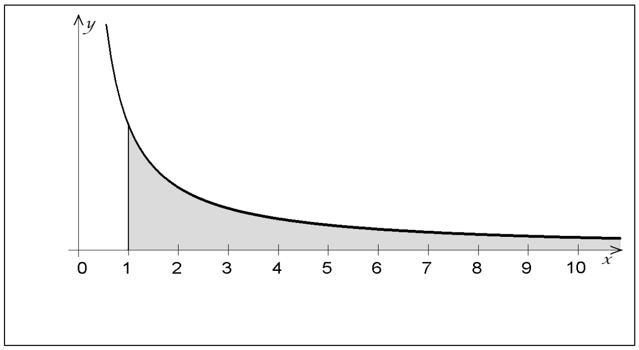
1.) ∞ 2.) 1 3.) $\frac{1}{5e^{10}}$ 4.) ∞ 5.) ∞ 6.) $\frac{3}{2}$ 7.) ∞ 8.) $-\infty$ 9.) $\frac{1}{4}$ 10.) $-\infty$

11.) $\frac{\pi\sqrt{3}}{9}$ 12.) ∞ 13.) $\frac{\pi}{2}$ 14.) 1

Sample Problems - Solutions

Infinite Limits of Integration

There are two types of improper integrals. The ones with infinite limits of integration are easy to recognize, we are asked about the area of a region that is infinitely long. For example, $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx$ and $\displaystyle\int_1^{\infty} \frac{1}{x^4}\, dx$ are such integrals. Let N be a very large positive number. The definite integral $\displaystyle\int_1^{N} \frac{1}{x^4}\, dx$ is defined for all positive N. So what we do is we let N approach infinity and determine what the values of the definite integrals are doing. If they approach a finite number, we define that to be the area under the graph. If the limit of the definite integrals is infinte, we say that the area under the graph is infinite, and the integral diverges.



1. $\displaystyle\int_1^{\infty} \frac{1}{x^4} dx = \frac{1}{3}$

Solution: We compute the limit of the definite integrals as the upper limit approaches infinity.

$$\int_1^{\infty} \frac{1}{x^4} dx = \lim_{N\to\infty} \int_1^{N} x^{-4} dx = \lim_{N\to\infty} \left.\frac{x^{-3}}{-3}\right|_1^N = \lim_{N\to\infty} \left(\frac{N^{-3}}{-3} - \frac{1^{-3}}{-3}\right) = \lim_{N\to\infty} \left(\underset{\substack{\downarrow \\ 0}}{\frac{-1}{3N^3}} - \left(\frac{-1}{3}\right)\right) = \frac{1}{3}$$

2. $\displaystyle\int_1^{\infty} \frac{1}{x} dx = \infty$

Solution: We compute the limit of the definite integrals as the upper limit approaches infinity.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{N\to\infty} \int_1^{N} \frac{1}{x} dx = \lim_{N\to\infty} \ln|x|\Big|_1^N = \lim_{N\to\infty} \left(\underset{\substack{\downarrow \\ \infty}}{\ln N} - \ln 1\right) = \infty$$

This improper integral diverges.

3. $\displaystyle\int_{10}^{\infty} \frac{1}{x\ln x} dx = \infty$

Solution: We first compute the indefinite integral by substitution. Let $u = \ln x$. Then $du = \frac{1}{x} dx$.

$$\int \frac{1}{x\ln x} dx = \int \frac{1}{\ln x}\left(\frac{1}{x} dx\right) = \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C$$

Now we are ready to evaluate the improper integral.

$$\int_{10}^{\infty} \frac{1}{x\ln x} dx = \lim_{N\to\infty} \int_{10}^{N} \frac{1}{x\ln x} dx = \lim_{N\to\infty} \ln\ln|x| \Big|_{10}^{N} = \lim_{N\to\infty} \Big(\underset{\substack{\downarrow \\ \infty}}{\ln(\ln N)} - \ln(\ln 10)\Big) = \infty$$

This improper integral diverges.

4. $\displaystyle\int_{0}^{\infty} e^{-5x} dx = \frac{1}{5}$

Solution:

$$\int_{0}^{\infty} e^{-5x} dx = \lim_{N\to\infty} \int_{0}^{N} e^{-5x} dx = \lim_{N\to\infty} \frac{e^{-5x}}{-5}\Big|_{0}^{N} = \lim_{N\to\infty} \left(\frac{e^{-5N}}{-5} - \frac{e^{-5(0)}}{-5}\right) = \lim_{N\to\infty} \left(\frac{-1}{\underset{\substack{\downarrow \\ 0}}{-5e^{5N}}} - \frac{1}{-5}\right) = \frac{1}{5}$$

5. $\displaystyle\int_{0}^{\infty} xe^{-x^2} dx = \frac{1}{2}$

Solution: We first compute the indefinite integral, by substitution. Let $u = -x^2$. Then $du = -2xdx$.

$$\int xe^{-x^2} dx = \int e^{-x^2}(xdx) = \int e^u\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

Now we are ready to evaluate the improper integral.

$$\int_{0}^{\infty} xe^{-x^2} dx = \lim_{N\to\infty} \int_{0}^{N} xe^{-x^2} dx = \lim_{N\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{0}^{N} = -\frac{1}{2}\lim_{N\to\infty} \frac{1}{e^{x^2}}\Big|_{0}^{N} = -\frac{1}{2}\lim_{N\to\infty} \left(\frac{1}{\underset{\substack{\downarrow \\ 0}}{e^{N^2}}} - \frac{1}{e^{0^2}}\right) = \frac{1}{2}$$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi$

Solution: Both limits of this integral are infinite. In case of such an integral, we separate it to a sum of two improper integrals

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx$$

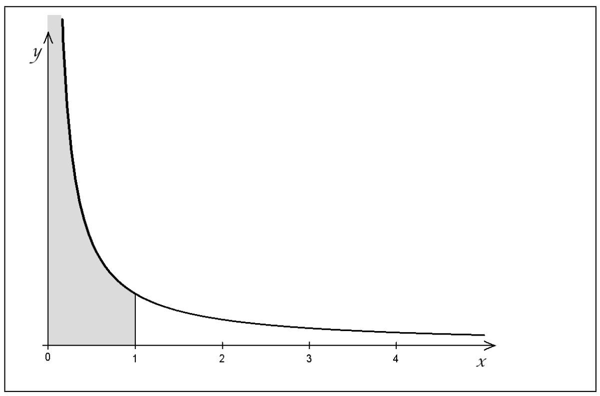
Because $f(x) = \dfrac{1}{1+x^2}$ is an even function, the two integrals are the same:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\lim_{N\to\infty}\int_{0}^{N} \frac{1}{1+x^2}\,dx$$

$$= 2\lim_{N\to\infty}\left(\tan^{-1} x\right)\Big|_0^N = 2\lim_{N\to\infty}\left(\tan^{-1} N - \tan^{-1} 0\right) = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

Integrands with Vertical Asymptotes

Some integrals are improper because they represent an infinitely tall region. These are not trickier to compute but more difficult to detect. For example, $\displaystyle\int_0^1 \frac{1}{x}dx$ appears to be a definite integral. However, there is a vertical asymptote at zero, making the integral improper.



7. $\displaystyle\int_0^1 \frac{1}{x}dx = \infty$

Solution: We compute the limit of definite integrals as the lower limit approaches zero.

$$\int_0^1 \frac{1}{x}dx = \lim_{h\to 0^+}\int_h^1 \frac{1}{x}dx = \lim_{h\to 0^+}\left(\ln|x|\right)\Big|_h^1 = \lim_{h\to 0^+}\left(\ln|1| - \ln|h|\right) = \lim_{h\to 0^+}\left(0 - (-\infty)\right) = \infty$$

The area under the graph is infinite; this integral diverges.

8. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx = 2$

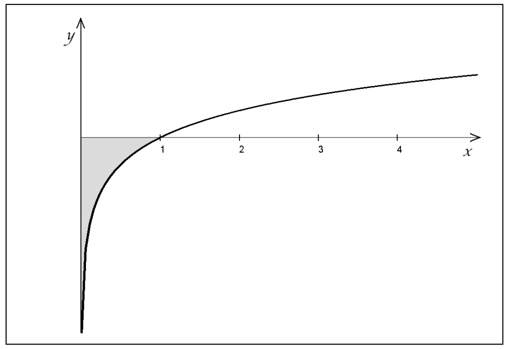
Solution: We compute the limit of definite integrals as the lower limit approaches zero.

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{h\to 0^+}\int_h^1 x^{-1/2}dx = \lim_{h\to 0^+}\left(2\sqrt{x}\right)\Big|_h^1 = \lim_{h\to 0^+}\left(2\sqrt{1} - 2\sqrt{h}\right) = 2$$

The area under the graph is 2.

9. $\displaystyle\int_0^1 \ln x \; dx = -1$

Solution: This is an improper integral because there is a vertical asymptote at zero.



We compute this integral by taking the limits of definite integrals, between a small positive number and 1.

$$\int_0^1 \ln x dx = \lim_{h\to 0^+} \int_h^1 \ln x dx$$

The antiderivative of $\ln x$ is $x \ln x - x$. We verified this fact by differentiation. (Later on, we will learn how to compute this integral). The improper integral is

$$\begin{aligned}\int_0^1 \ln x dx &= \lim_{h\to 0^+} \int_h^1 \ln x dx = \lim_{h\to 0^+} (x\ln x - x)\Big|_h^1 = \lim_{h\to 0^+} ((1\ln 1 - 1) - (h\ln h - h)) \\ &= \lim_{h\to 0^+} \left(-1 - \underset{?}{\underset{\downarrow}{h\ln h}} - \underset{0}{\underset{\downarrow}{h}}\right)\end{aligned}$$

As h approaches zero from the right, $\ln h$ approaches negative infinity. This means that $\displaystyle\lim_{h\to 0^+} h\ln h = \lim_{h\to 0^+} \frac{\ln h}{\frac{1}{h}}$ is an $\dfrac{\infty}{\infty}$ type of an indeterminate. We apply L'Hôpital's rule:

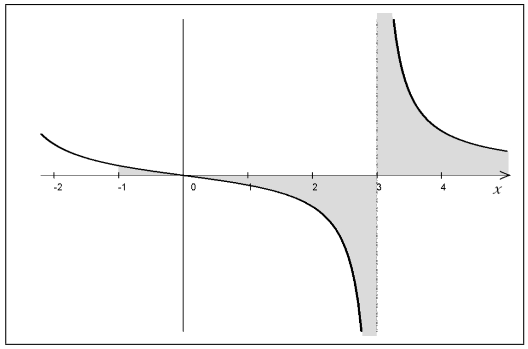
$$\lim_{h\to 0^+} \frac{\ln h}{\frac{1}{h}} = \lim_{h\to 0^+} \frac{\frac{1}{h}}{-\frac{1}{h^2}} = \lim_{h\to 0^+} \frac{1}{h}\left(\frac{-h^2}{1}\right) = \lim_{h\to 0^+} (-h) = 0$$

Thus the improper integral is

$$\int_0^1 \ln x dx = \lim_{h\to 0^+} \left(-1 - \underset{0}{\underset{\downarrow}{h\ln h}} - \underset{0}{\underset{\downarrow}{h}}\right) = -1$$

10. $\displaystyle\int_{-1}^{4} \frac{x}{x^2-9}\,dx = \text{undefined}$

Solution: This is an imporper integral because there is a vertical asymptote at $x = 3$.



We again separate this into two improper integrals, the area of the region to the left of 3 and to the right of 3.

$$\int_{-1}^{4} \frac{x}{x^2-9}dx = \int_{-1}^{3} \frac{x}{x^2-9}dx + \int_{3}^{4} \frac{x}{x^2-9}dx = \lim_{a\to 3^-}\int_{-1}^{a} \frac{x}{x^2-9}dx + \lim_{b\to 3^+}\int_{b}^{4} \frac{x}{x^2-9}dx$$

We first compute the indefinite integral by substitution. Let $u = x^2 - 9$. Then $du = 2xdx$.

$$\int \frac{x}{x^2-9}dx = \int \frac{1}{x^2-9}(xdx) = \int \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2-9\right| + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_{-1}^{3} \frac{x}{x^2-9}dx &= \lim_{a\to 3^-}\int_{-1}^{a} \frac{x}{x^2-9}dx = \lim_{a\to 3^-}\frac{1}{2}\ln\left|x^2-9\right|\Big|_{-1}^{a} = \frac{1}{2}\lim_{a\to 3^-}\ln\left|x^2-9\right|\Big|_{-1}^{a} \\ &= \frac{1}{2}\lim_{a\to 3^-}\left(\ln\left|a^2-9\right| - \ln\left|(-1)^2-9\right|\right) = \frac{1}{2}\lim_{a\to 3^-}\left(\ln\left|a^2-9\right| - \ln 8\right) = -\infty\end{aligned}$$

The other part:

$$\begin{aligned}\int_{3}^{4} \frac{x}{x^2-9}dx &= \lim_{b\to 3^+}\int_{b}^{4} \frac{x}{x^2-9}dx = \lim_{b\to 3^+}\frac{1}{2}\ln\left|x^2-9\right|\Big|_{b}^{4} = \lim_{b\to 3^+}\left(\frac{1}{2}\ln\left|4^2-9\right| - \frac{1}{2}\ln\left|b^2-9\right|\right) \\ &= \lim_{b\to 3^+}\left(\frac{1}{2}\ln 7 - \frac{1}{2}\ln\left|b^2-9\right|\right) = \infty\end{aligned}$$

When we combine two improper integrals, finite sums are allowed to be added, such as in problem #7. However, the sum $\infty + (-\infty)$ is an indeterminate and in this case, it is not defined. This integral is undefined. We also say that the area under the graph is undefined.

11. $\displaystyle\int_{0}^{25} \frac{1}{\sqrt{x}}\,dx = 10$

Solution:

$$\int_{0}^{25} \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+}\int_{a}^{25} x^{-1/2}dx = \lim_{a\to 0^+} 2\sqrt{x}\Big|_{a}^{25} = 2\lim_{a\to 0^+}\sqrt{x}\Big|_{a}^{25} = 2\lim_{a\to 0^+}\left(\sqrt{25}-\sqrt{a}\right) = 10$$

12. $\displaystyle\int_1^2 \frac{1}{\sqrt{2-x}}dx = 2$

Solution: We first compute the indefinite integral, by substitution. Let $u = 2-x$. Then $du = -dx$ and so $dx = -du$.

$$\int \frac{1}{\sqrt{2-x}}dx = \int \frac{1}{\sqrt{u}}(-du) = -\int u^{-1/2}du = -2u^{1/2} + C = -2\sqrt{2-x} + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{2-x}}dx &= \lim_{a\to 2^-}\int_1^a \frac{1}{\sqrt{2-x}}dx = \lim_{a\to 2^-} -2\sqrt{2-x}\Big|_1^a = -2\lim_{a\to 2^-}\sqrt{2-x}\Big|_1^a \\ &= -2\lim_{a\to 2^-}\left(\sqrt{2-a}-\sqrt{2-1}\right) = -2\lim_{a\to 2^-}\left(\sqrt{2-2}-\sqrt{1}\right) = -2(-1) = 2\end{aligned}$$

13. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{2-x}}dx = -\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$

Solution: We first compute the indefinite integral, by substitution. Let $u = 2-x$. Then $du = -dx$ and so $dx = -du$

$$\int \frac{1}{\sqrt[3]{2-x}} = \int \frac{1}{\sqrt[3]{u}}(-du) = -\int u^{-1/3}du = -\frac{3}{2}u^{2/3} + C = -\frac{3}{2}(2-x)^{2/3} + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_0^5 \frac{1}{\sqrt[3]{2-x}}dx &= \int_0^2 \frac{1}{\sqrt[3]{2-x}}dx + \int_2^5 \frac{1}{\sqrt[3]{2-x}}dx = \lim_{a\to 2^-}\int_0^a \frac{1}{\sqrt[3]{2-x}}dx + \lim_{b\to 2^+}\int_b^5 \frac{1}{\sqrt[3]{2-x}}dx \\ &= \lim_{a\to 2^-} -\frac{3}{2}(2-x)^{2/3}\Big|_0^a + \lim_{b\to 2^+} -\frac{3}{2}(2-x)^{2/3}\Big|_b^5 = -\frac{3}{2}\left(\lim_{a\to 2^-}(2-x)^{2/3}\Big|_0^a + \lim_{b\to 2^+}(2-x)^{2/3}\Big|_b^5\right) \\ &= -\frac{3}{2}\left(\lim_{a\to 2^-}\left((2-a)^{2/3}-(2-0)^{2/3}\right) + \lim_{b\to 2^+}\left((2-5)^{2/3}-(2-b)^{2/3}\right)\right) \\ &= -\frac{3}{2}\left(-2^{2/3}+(-3)^{2/3}\right) = -\frac{3}{2}\left(-\sqrt[3]{4}+\sqrt[3]{9}\right) = -\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)\end{aligned}$$

