Sample Problems

1. Compute the trapezoidal approximation for $\int_0^2 \sqrt{x}dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

2. Use Simpson's rule to approximate $\int_0^2 \sqrt{x}dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

Practice Problems

1. a) Compute the trapezoidal approximation for $\int_0^2 \sqrt{x}\, dx$ using a regular partition with $n = 6$. Compare the estimate with the exact value.

   b) Use Simpson's rule to approximate $\int_0^2 \sqrt{x}\, dx$ using a regular partition with $n = 6$. Compare the estimate with the exact value.

2. a) Compute the trapezoidal approximation for $\int_0^1 e^{-x^2} dx$ using a regular partition with $n = 10$.

   b) Use Simpson's rule to approximate $\int_0^1 e^{-x^2} dx$ using a regular partition with $n = 10$.

3. a) Compute the trapezoidal approximation for $\int_0^1 \frac{1}{1+x^2} dx$ using a regular partition with $n = 8$. Compare the estimate with the exact value.

   b) Use Simpson's rule to approximate $\int_0^1 \frac{1}{1+x^2} dx$ using a regular partition with $n = 8$. Compare the estimate with the exact value.

Sample Problems - Answers

1. $T = \frac{1}{4}\left(2 + 2\sqrt{2} + \sqrt{6}\right) \approx 1.819\,48$ The exact value of the integral is $\frac{4\sqrt{2}}{3} \approx 1.885\,62$
   The approximation underestimates the actual area with an error of 0.035076 or 3.51% of the exact value.

2. $\frac{1}{6}\left(3\sqrt{2} + 2\sqrt{6} + 2\right) \approx 1.856\,936\,7$ The exact value of the integral is $\frac{4\sqrt{2}}{3} \approx 1.885\,62$
   The approximation underestimates the actual area with an error of 0.01521 16 or 1.52% of the exact value.

Practice Problems - Answers

1. a) $T \approx 1.848\,88$ underestimates the area with an error of $0.019\,483\,310\,8$ or 1.948% of the area
   b) $P \approx 1.869\,998\,52$ underestimates the area with an error of 0.008284 or 0.83% of the area.

2. a) $T \approx 0.746\,210\,796\,131\,749$ b) $P \approx 0.746\,824\,948\,254\,443$

3. a) $T \approx 0.784\,747\,1$ $\displaystyle\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4} \approx 0.785\,3982$

   error: $0.000\,651\,1$ which is 0.000829 or 0.083% of the exact value

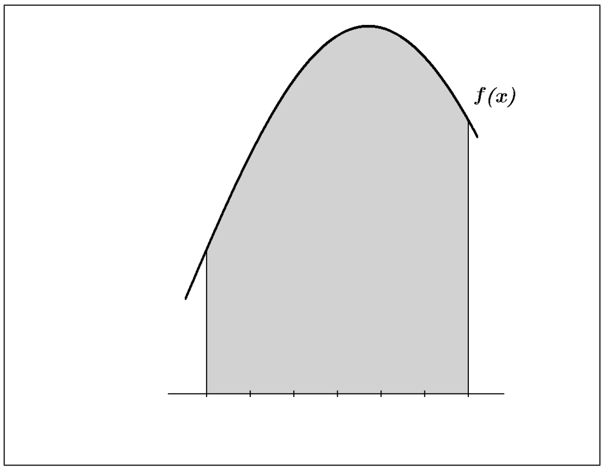
   b) $P \approx 0.785\,398\,125\,614\,677$ $\displaystyle\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4} \approx 0.785\,398\,163\,397\,448$

   error is less than 3.78×10^{-8} which is 4.811×10^{-6} % of the area

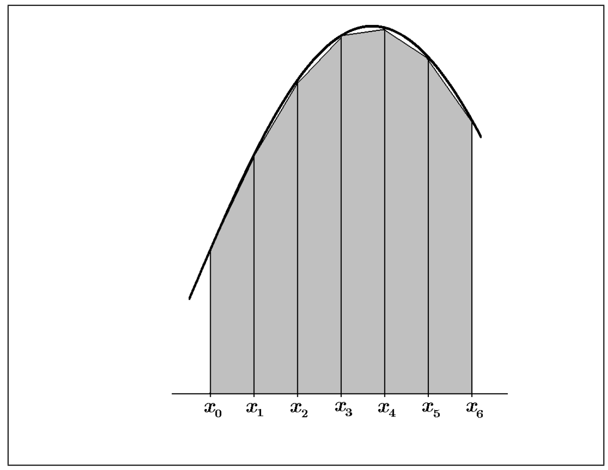
Sample Problems - Solutions

Trapezoidal Approximations

Consider $f(x)$ on the interval $[a,b]$. Define $\Delta x = \dfrac{b-a}{n}$, in other words, we are looking at a regular partition of $[a,b]$ into n subintervals. In the trapezoidal approximation, the area under the function is approximated by a sum of areas of trapezoids. The picture below shows the trapezoidal approximation of the area under a graph $f(x)$, using a regular partition with $n=6$.



area under the graph



trapezoidal approximation

Recall that a trapezoid's area is $A = \dfrac{1}{2}(a+c)\,h$ where a and c are a pair of parallel sides and h is their distance. Based on our regular partition of $n=6$, the area of the first trapezoid is $A_1 = \dfrac{1}{2}\left(f(x_0)+f(x_1)\right)\Delta x$, the area of the second trapezoid is $A_2 = \dfrac{1}{2}\left(f(x_1)+f(x_2)\right)\Delta x$, and so on, the area of the last trapezoid is $A_6 = \dfrac{1}{2}\left(f(x_1)+f(x_2)\right)\Delta x$. So the trapezoidal approximation is

$$T = \frac{1}{2}\left(f(x_0)+f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1)+f(x_2)\right)\Delta x + \frac{1}{2}\left(f(x_2)+f(x_3)\right)\Delta x + ... + \frac{1}{2}\left(f(x_5)+f(x_6)\right)\Delta x$$

After we factor out $\dfrac{\Delta x}{2}$, we have

$$T = \frac{\Delta x}{2}\ \left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right)$$

where the values of x_k can be computed as follows: recall that $\Delta x = \dfrac{b-a}{6}$ and

$$x_0 = a,\ \ x_1 = a+\Delta x,\ \ x_2 = a+2\Delta x,\ \ x_3 = a+3\Delta x,\quad \text{in general,}\ \ x_k = a+k\Delta x$$

Example 1: Compute the trapezoidal approximation for $\displaystyle\int_0^2 \sqrt{x}\,dx$ using a regular partition with $n=4$. Compare the estimate with the exact value.

Solution: The interval is 2 units long, so with $n=4$, each subinterval will have length $\dfrac{1}{2}$. Then

$$x_0 = 0 = \frac{0}{2},\ \ x_1 = \frac{1}{2},\ \ x_2 = \frac{2}{2},\ \ x_3 = \frac{3}{2},\ \ \text{and}\ \ x_4 = \frac{4}{2} = 2$$

The trapezoidal approximation is

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right)=\frac{1}{4}\left(\sqrt{0}+2\sqrt{\frac{1}{2}}+2\sqrt{1}+2\sqrt{\frac{3}{2}}+\sqrt{2}\right) \\ &= \frac{1}{4}\left(0+\sqrt{2}+2+\sqrt{6}+\sqrt{2}\right)=\frac{1}{4}\left(2+2\sqrt{2}+\sqrt{6}\right)\approx 1.819\,48 \end{aligned}$$

The exact value of the integral is

$$\int_0^2 \sqrt{x}dx=\int_0^2 x^{1/2}dx=\frac{2}{3}x^{3/2}\Big|_0^2=\frac{2}{3}2^{3/2}-\frac{2}{3}0^{3/2}=\frac{2}{3}\left(2\sqrt{2}\right)=\frac{4\sqrt{2}}{3}\approx 1.885\,62$$

The approximation underestimates the actual area, the error is $1.885\,62-1.819\,48=0.066\,14$ and that is $\frac{0.066\,14}{1.885\,62}\approx 0.035076$ or 3.51% of the exact value.

Approximations using parabolas - Simpson's Rule

With this approximation, the regular partition must have an even number of subintervals. in other words, n must be even. We approximate the area under the graph over two consecutive subintervals as the area under the parabola connecting the three endpoints in the two subintervals.

Let (x_{i-1},y_{i-1}) (x_i,y_i) and (x_{i+1},y_{i+1}) will be the three points given to be on the parabola $y=Ax^2+Bx+C$. We may assume that $x_i=0$. This is because the area of the parabola does not change if we horizontally shift it until x_i is shifted into the origin. Let h denote the length of the subintervals. In other words, $h=\frac{b-a}{n}$. With this notation, the parabola must pass through the points $(-h,y_{i-1})$ $(0,y_i)$ and (h,y_{i+1}).

We need to solve for A, B, and C, in terms of x_i,h and y_i. We will have three equations, stating that each point is on the parabola.

$$\begin{aligned} y_{i-1} &= A(-h)^2+B(-h)+C \\ y_i &= A\left(0^2\right)+B(0)+C \\ y_{i+1} &= A(h)^2+B(h)+C \end{aligned}$$

$$\begin{aligned} y_{i-1} &= Ah^2-Bh+C \\ y_i &= C \\ y_{i+1} &= Ah^2+Bh+C \end{aligned}$$

Let us first eliminate the second equation by substituting y_i for C.

$$\begin{aligned} y_{i-1} &= Ah^2-Bh+y_i \\ y_{i+1} &= Ah^2+Bh+y_i \end{aligned}$$

$$\begin{aligned} y_{i-1}-y_i &= Ah^2-Bh \\ y_{i+1}-y_i &= Ah^2+Bh \end{aligned}$$

Add the two equations to get A:

$$\begin{aligned} y_{i-1}-y_i+y_{i+1}-y_i &= 2Ah^2 \\ \frac{1}{2h^2}\left(y_{i-1}-2y_i+y_{i+1}\right) &= A \end{aligned}$$

Subtract the two equations to get B:

$$\begin{aligned} y_{i+1} - y_i - (y_{i-1} - y_i) &= 2Bh \\ y_{i+1} - y_i - y_{i-1} + y_i &= 2Bh \\ \frac{1}{2h}(y_{i+1} - y_{i-1}) &= B \end{aligned}$$

Thus we have the parabola: $A = \frac{1}{2h^2}(y_{i-1} - 2y_i + y_{i+1})$, $B = \frac{1}{2h}(y_{i+1} - y_{i-1})$, and $C = y_i$.

Now the the area under the parabola is

$$\begin{aligned} P &= \int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{-h}^{h} = \left(\frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch\right) - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h)\right) \\ &= \left(\frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch\right) - \left(-\frac{A}{3}h^3 + \frac{B}{2}h^2 - Ch\right) = \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\ &= \frac{2}{3}Ah^3 + 2Ch \\ &= \frac{2}{3}\left(\frac{1}{2h^2}(y_{i-1} - 2y_i + y_{i+1})\right)h^3 + 2(y_i)h \\ &= \frac{1}{3}(y_{i-1} - 2y_i + y_{i+1})h + 2(y_i)h \\ &= \frac{h}{3}(y_{i-1} - 2y_i + y_{i+1} + 6y_i) = \frac{h}{3}(y_{i-1} + 4y_i + y_{i+1}) \end{aligned}$$

The first parabola has area $A_1 = \frac{h}{3}(y_0 + 4y_1 + y_2)$ The second parabola will begin at the same point where the previous one ends, namely in point (x_2, y_2), Thus $A_2 = \frac{h}{3}(y_2 + 4y_3 + y_4)$. The third parabola has area $A_3 = \frac{h}{3}(y_4 + 4y_5 + y_6)$ And so on, the last parabolaa has area $A_k = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$. As we add these areas, the approximation is

$$P = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + ... + 2y_{n-2} + 4y_{n-1} + y_n)$$

The coefficients are: $1, 4, 2, 4, 2, 4, 2, 4, ...2, 4, 1$. The same fact can be expressed using the more traditional notation:

$$P = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

Example 2: Use Simpson's rule to approximate $\int_0^2 \sqrt{x}dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

Solution: The interval is 2 units long, so with $n = 4$, each subinterval will have length $\frac{1}{2}$. Then

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 2, \quad x_3 = \frac{3}{2}, \quad \text{and} \quad x_4 = 2$$

The approximation with Simpson's rule is

$$\begin{aligned} P &= \frac{\Delta x}{3}\left(f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+f\left(x_4\right)\right)=\frac{1}{6}\left(\sqrt{0}+4\sqrt{\frac{1}{2}}+2\sqrt{1}+4\sqrt{\frac{3}{2}}+\sqrt{2}\right) \\ &= \frac{1}{6}\left(0+2\sqrt{2}+2+2\sqrt{6}+\sqrt{2}\right)=\frac{1}{6}\left(3\sqrt{2}+2\sqrt{6}+2\right)\approx 1.8569367 \end{aligned}$$

We already computed the exact value of the integral $\int\limits_0^2 \sqrt{x}dx = \frac{4\sqrt{2}}{3} \approx 1.88562$

The approximation underestimates the actual area, the error is $1.88562 - 1.8569367 = 0.0286833$ and that is $\frac{0.0286833}{1.88562} = 0.01521$ or 1.52% of the exact value.

