Sample Problems

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$
2. $\displaystyle\int \frac{2x}{(x+3)(3x+1)}\,dx$
3. $\displaystyle\int \frac{x+5}{x^2-2x-3}\,dx$
4. $\displaystyle\int \frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4}\,dx$
5. $\displaystyle\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx$
6. $\displaystyle\int \frac{2x-1}{(x-5)^2}\,dx$
7. $\displaystyle\int \frac{x+3}{(x-1)^3}\,dx$
8. $\displaystyle\int \frac{x^4}{x^4-1}\,dx$
9. $\displaystyle\int \sec x\,dx$
10. $\displaystyle\int \operatorname{csch} x\,dx$

Practice Problems

1. $\displaystyle\int \frac{1}{x^2+3x}\,dx$
2. $\displaystyle\int \frac{x-5}{x^2-2x-8}\,dx$
3. $\displaystyle\int \frac{1}{x^2-a^2}\,dx$
4. $\displaystyle\int \frac{x-1}{x^2-4}\,dx$
5. $\displaystyle\int \frac{x-1}{x^2+4}\,dx$
6. $\displaystyle\int \frac{x^2}{x^2+2x-3}\,dx$
7. $\displaystyle\int \frac{2x^3-x^2-10x-4}{x^2-4}\,dx$
8. $\displaystyle\int \frac{5x-17}{x^2-6x+9}\,dx$
9. $\displaystyle\int \frac{2x^2+7x+3}{x^2+1}\,dx$
10. $\displaystyle\int \frac{2x^2-x+20}{(x-2)(x^2+9)}\,dx$
11. $\displaystyle\int \frac{x^4}{x^4-16}\,dx$
12. $\displaystyle\int \frac{2x+1}{x^2+1}\,dx$
13. $\displaystyle\int \frac{x^2+2}{x(x^2+6)}\,dx$
14. $\displaystyle\int \frac{-x+6}{(x+3)^2}\,dx$
15. $\displaystyle\int \frac{2x-3}{x^2+9}\,dx$
16. $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

Sample Problems - Answers

1.) $\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C$ 2.) $\frac{3}{4}\ln|x+3|-\frac{1}{12}\ln|3x+1|+C$ 3.) $2\ln|x-3|-\ln|x+1|+C$

4.) $\frac{x^3}{3}-x^2+5x+\frac{13}{5}\ln|x+4|+\frac{2}{5}\ln|x-1|+C$ 5.) $-\frac{1}{6}\ln|x+1|-\frac{1}{3}\ln|x-2|+\frac{2}{3}\ln|x-5|+C$

6.) $2\ln|x-5|-\frac{9}{x-5}+C$ 7.) $-\frac{1}{x-1}-\frac{2}{(x-1)^2}+C=\frac{-x-1}{(x-1)^2}+C$

8.) $x-\frac{1}{2}\arctan x-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|+C$ 9.) $\ln|\sec x+\tan x|+C=-\ln|\sec x-\tan x|+C$

10.) $\ln|e^x-1|-\ln(e^x+1)+C$

Practice Problems - Answers

1.) $\frac{1}{3}\ln|x|-\frac{1}{3}\ln|x+3|+C$ 2.) $\frac{7}{6}\ln|x+2|-\frac{1}{6}\ln|x-4|+C$ 3.) $\frac{1}{2a}\ln|x-a|-\frac{1}{2a}\ln|x+a|+C$

4.) $\frac{1}{4}\ln|x-2|+\frac{3}{4}\ln|x+2|+C$ 5.) $\frac{1}{2}\ln\left(x^2+4\right)-\frac{1}{2}\tan^{-1}\frac{1}{2}x+C$

6.) $x+\frac{1}{4}\ln|x-1|-\frac{9}{4}\ln|x+3|+C$ 7.) $x^2-x-3\ln|x-2|+\ln|x+2|+C$

8.) $5\ln|x-3|+\frac{2}{x-3}+C$ 9.) $2x+\frac{7}{2}\ln\left(x^2+1\right)+\tan^{-1}x+C$

10.) $2\ln|x-2|-\frac{1}{3}\tan^{-1}\frac{x}{3}+C$ 11.) $x+\frac{1}{2}\ln|x-2|-\frac{1}{2}\ln|x+2|-\tan^{-1}\frac{x}{2}+C$

12.) $\tan^{-1}x+\ln\left(x^2+1\right)+C$ 13.) $\frac{1}{3}\ln\left|x^3+6x\right|+C$ 14.) $-\ln|x+3|-\frac{9}{x+3}+C$

15.) $\ln\left(x^2+9\right)-\tan^{-1}\frac{x}{3}+C$ 16.) $\ln|x|+\ln|x-1|-\ln|x+1|+C$

Sample Problems - Solutions

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$

Solution: We factor the denominator: $x^2-4=(x+2)(x-2)$. Next, we re-write the fraction $\dfrac{1}{x^2-4}$ as a sum (or difference) of fractions with denominators $x+2$ and $x-2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+2}+\frac{B}{x-2}=\frac{1}{x^2-4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-2)}{(x+2)(x-2)}+\frac{B(x+2)}{(x-2)(x+2)}=\frac{Ax-2A+Bx+2B}{x^2-4}=\frac{(A+B)x-2A+2B}{x^2-4}$$

Thus we have

$$\frac{(A+B)x-2A+2B}{x^2-4}=\frac{1}{x^2-4}$$

We clear the denominators by multiplication

$$(A+B)x-2A+2B=1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. In other words,

$$(A+B)x-2A+2B=0x+1$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A+B &= 0\\ -2A+2B &= 1 \end{aligned}$$

We solve this system and obtain $A=-\dfrac{1}{4}$ and $B=\dfrac{1}{4}$.

So our fraction, $\dfrac{1}{x^2-4}$ can be re-written as $\dfrac{-\frac{1}{4}}{x+2}+\dfrac{\frac{1}{4}}{x-2}$. We check:

$$\begin{aligned}\frac{-\frac{1}{4}}{x+2}+\frac{\frac{1}{4}}{x-2} &= \frac{-\frac{1}{4}(x-2)}{(x+2)(x-2)}+\frac{\frac{1}{4}(x+2)}{(x-2)(x+2)}=\frac{-\frac{1}{4}(x-2)+\frac{1}{4}(x+2)}{(x+2)(x-2)}=\frac{-\frac{1}{4}x+\frac{1}{2}+\frac{1}{4}x+\frac{1}{2}}{(x+2)(x-2)}\\ &= \frac{1}{x^2-4}\end{aligned}$$

Now we can easily integrate:

$$\int\frac{1}{x^2-4}\,dx=\int\frac{-\frac{1}{4}}{x+2}+\frac{\frac{1}{4}}{x-2}\,dx=-\frac{1}{4}\int\frac{1}{x+2}\,dx+\frac{1}{4}\int\frac{1}{x-2}\,dx=\boxed{-\frac{1}{4}\ln|x+2|+\frac{1}{4}\ln|x-2|+C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+2}+\frac{B}{x-2}=\frac{1}{x^2-4}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)}{(x+2)(x-2)}+\frac{B(x+2)}{(x-2)(x+2)}=\frac{1}{x^2-4}$$

and then multiply both sides by the denominator:

$$A(x-2)+B(x+2)=1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=2$ into both sides:

$$\begin{aligned} A(0)+B(4) &= 1 \\ B &= \frac{1}{4} \end{aligned}$$

Let us substitute $x=-2$ into both sides:

$$\begin{aligned} A(-4)+B(0) &= 1 \\ A &= -\frac{1}{4} \end{aligned}$$

and so $A=-\frac{1}{4}$ and $B=\frac{1}{4}$.

2. $\displaystyle\int \frac{2x}{(x+3)(3x+1)}\,dx$

Solution: We re-write the fraction $\dfrac{2x}{(x+3)(3x+1)}$ as a sum (or difference) of fractions with denominators $x+3$ and $3x+1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+3}+\frac{B}{3x+1}=\frac{2x}{(x+3)(3x+1)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x+3}+\frac{B}{3x+1} &= \frac{A(3x+1)}{(x+3)(3x+1)}+\frac{B(x+3)}{(x+3)(3x+1)}=\frac{A(3x+1)+B(x+3)}{(x+3)(3x+1)}=\frac{3Ax+A+Bx+3B}{(x+3)(3x+1)} \\ &= \frac{(3A+B)x+A+3B}{(x+3)(3x+1)} \end{aligned}$$

Thus we have

$$\frac{(3A+B)x+A+3B}{(x+3)(3x+1)}=\frac{2x}{(x+3)(3x+1)}$$

We clear the denominators by multiplication

$$(3A+B)x+A+3B=2x$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. In other words,

$$(3A+B)x+A+3B=2x+0$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} 3A + B &= 2 \\ A + 3B &= 0 \end{aligned}$$

We solve this system and obtain $A = \frac{3}{4}$ and $B = -\frac{1}{4}$.

So our fraction, $\frac{2x}{(x+3)(3x+1)}$ can be re-written as $\frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1}$. We check:

$$\begin{aligned} \frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1} &= \frac{\frac{3}{4}(3x+1)}{(x+3)(3x+1)} + \frac{-\frac{1}{4}(x+3)}{(x+3)(3x+1)} = \frac{\frac{3}{4}(3x+1) - \frac{1}{4}(x+3)}{(x+3)(3x+1)} = \frac{\frac{9}{4}x + \frac{3}{4} - \frac{1}{4}x - \frac{3}{4}}{(x+3)(3x+1)} \\ &= \frac{2x}{(x+3)(3x+1)} \end{aligned}$$

Now we can easily integrate:

$$\int \frac{2x}{(x+3)(3x+1)}\,dx = \int \frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1}\,dx = \frac{3}{4}\int \frac{1}{x+3}\,dx - \frac{1}{4}\int \frac{1}{3x+1}\,dx = \boxed{\frac{3}{4}\ln|x+3| - \frac{1}{12}\ln|3x+1| + C}$$

The second integral can be computed using the substitution $u = 3x + 1$.

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+3} + \frac{B}{3x+1} = \frac{2x}{(x+3)(3x+1)}$$

We bring the fractions to the common denominator:

$$\frac{A(3x+1)}{(x+3)(3x+1)} + \frac{B(x+3)}{(x+3)(3x+1)} = \frac{2x}{(x+3)(3x+1)}$$

and then multiply both sides by the denominator:

$$A(3x+1) + B(x+3) = 2x$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x = -\frac{1}{3}$ into both sides:

$$\begin{aligned} A(0) + B\left(-\frac{1}{3} + 3\right) &= 2\left(-\frac{1}{3}\right) \\ \frac{8}{3}B &= -\frac{2}{3} \\ B &= -\frac{1}{4} \end{aligned}$$

Let us substitute $x = -3$ into both sides:

$$\begin{aligned} A(3(-3)+1) + B(-3+3) &= 2(-3) \\ -8A + B(0) &= -6 \\ -8A &= -6 \\ A &= \frac{3}{4} \end{aligned}$$

and so $A = \frac{3}{4}$ and $B = -\frac{1}{4}$.

3. $\displaystyle\int \frac{x+5}{x^2-2x-3}\,dx$

Solution: We factor the denominator: $x^2-2x-3=(x+1)(x-3)$. Next, we re-write the fraction $\dfrac{x+5}{x^2-2x-3}$ as a sum (or difference) of fractions with denominators $x+1$ and $x-3$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+1}+\frac{B}{x-3}=\frac{x+5}{x^2-2x-3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x+1)(x-3)}+\frac{B(x+1)}{(x-3)(x+1)}=\frac{Ax-3A+Bx+B}{x^2-2x-3}=\frac{(A+B)x-3A+B}{x^2-2x-3}$$

Thus

$$\frac{(A+B)x-3A+B}{x^2-2x-3}=\frac{x+5}{x^2-2x-3}$$

We clear the denominators by multiplication

$$(A+B)x-3A+B=x+5$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A+B &= 1\\ -3A+B &= 5 \end{aligned}$$

We solve the system and obtain $A=-1$ and $B=2$.

So we have that our fraction, $\dfrac{x+5}{x^2-2x-3}$ can be re-written as $\dfrac{-1}{x+1}+\dfrac{2}{x-3}$. We check:

$$\frac{-1}{x+1}+\frac{2}{x-3}=\frac{-1(x-3)}{(x+1)(x-3)}+\frac{2(x+1)}{(x-3)(x+1)}=\frac{-(x-3)+2(x+1)}{(x+1)(x-3)}=\frac{-x+3+2x+2}{x^2-2x-3}=\frac{x+5}{x^2-2x-3}$$

Now we can easily integrate:

$$\int\frac{x+5}{x^2-2x-3}\,dx=\int\frac{-1}{x+1}+\frac{2}{x-3}\,dx=-\int\frac{1}{x+1}\,dx+2\int\frac{1}{x-3}\,dx=\boxed{-\ln|x+1|+2\ln|x-3|+C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1}+\frac{B}{x-3}=\frac{x+5}{x^2-2x-3}$$

We bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x-3)(x+1)}+\frac{B(x+1)}{(x-3)(x+1)}=\frac{x+5}{x^2-2x-3}$$

and then multiply both sides by the denominator:

$$A(x-3)+B(x+1)=x+5$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=3$ into both sides:

$$\begin{aligned} A(0)+B(4) &= 3+5\\ 4B &= 8\\ B &= 2 \end{aligned}$$

Let us substitute $x = -1$ into both sides:

$$\begin{aligned} A(-4) + B(0) &= -1 + 5 \\ -4A &= 4 \\ A &= -1 \end{aligned}$$

and so $A = -1$ and $B = 2$.

4. $\displaystyle\int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4}\, dx$

Solution: This rational function is an improper fraction since the numerator has a higher degree than the denominator. We first perform long division. This process is similar to long division among numbers. For example, to simplify $\frac{38}{7}$, we perform the long division $38 \div 7 = 5$ R 3 which is the same thing as to say that $\frac{38}{7} = 5\frac{3}{7}$. The division:

$$\begin{array}{rrrrrrrrrrrrr}
 & & & & & x^2 & - & 2x & + & 5 & & & \\ \cline{6-13}
x^2 & + & 3x & - & 4 \;) & x^4 & + & x^3 & - & 5x^2 & + & 26x & - \; 21 \\
 & & & & & -x^4 & - & 3x^3 & + & 4x^2 & & & \\ \cline{6-13}
 & & & & & & - & 2x^3 & - & x^2 & + & 26x & - \; 21 \\
 & & & & & & & 2x^3 & + & 6x^2 & - & 8x & \\ \cline{8-13}
 & & & & & & & & & 5x^2 & + & 18x & - \; 21 \\
 & & & & & & & & - & 5x^2 & - & 15x & + \; 20 \\ \cline{9-13}
 & & & & & & & & & & & 3x & - \; 1
\end{array}$$

Step 1: $\frac{x^4}{x^2} = x^2$

$x^2\left(x^2 + 3x - 4\right) = x^4 + 3x^3 - 4x^2$

$-\left(x^4 + 3x^3 - 4x^2\right) = -x^4 - 3x^3 + 4x^2$

We add that to the original polynomial shown above.

Step 2: $\frac{-2x^3}{x^2} = -2x$

$-2x\left(x^2 + 3x - 4\right) = -2x^3 - 6x^2 + 8x$

$-1\left(-2x^3 - 6x^2 + 8x\right) = 2x^3 + 6x^2 - 8x$

We add that to the original polynomial shown above.

Step 3: $\frac{5x^2}{x^2} = 5$

$5\left(x^2 + 3x - 4\right) = 5x^2 + 15x - 20$

$-1\left(5x^2 + 15x - 20\right) = -5x^2 - 15x + 20$

We add that to the original polynomial shown above.

The result of this computation is that

$$\frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} = x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4}$$

very much like $\frac{38}{7} = 5 + \frac{3}{7}$. Thus

$$\begin{aligned} \int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4}\, dx &= \int x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4}\, dx = \int x^2 - 2x + 5\, dx + \int \frac{3x - 1}{x^2 + 3x - 4}\, dx \\ &= \frac{x^3}{3} - x^2 + 5x + C_1 + \int \frac{3x - 1}{x^2 + 3x - 4}\, dx \end{aligned}$$

We apply the method of partial fractions to compute $\displaystyle\int \frac{3x - 1}{x^2 + 3x - 4}\, dx$.

We factor the denominator: $x^2+3x-4=(x+4)(x-1)$. Next, we re-write the fraction $\dfrac{3x-1}{x^2+3x-4}$ as a sum (or difference) of fractions with denominators $x+4$ and $x-1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+4}+\frac{B}{x-1}=\frac{3x-1}{x^2+3x-4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-1)}{(x+4)(x-1)}+\frac{B(x+4)}{(x+4)(x-1)}=\frac{Ax-A+Bx+4B}{x^2+3x-4}=\frac{(A+B)x-A+4B}{x^2+3x-4}$$

Thus

$$\frac{(A+B)x-A+4B}{x^2+3x-4}=\frac{3x-1}{x^2+3x-4}$$

We clear the denominators by multiplication

$$(A+B)x-A+4B=3x-1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{aligned} A+B &= 3 \\ -A+4B &= -1 \end{aligned}$$

We solve the system and obtain $A=\dfrac{13}{5}$ and $B=\dfrac{2}{5}$.

So our fraction, $\dfrac{3x-1}{x^2+3x-4}$ can be re-written as $\dfrac{\frac{13}{5}}{x+4}+\dfrac{\frac{2}{5}}{x-1}$. We check:

$$\begin{aligned} \frac{\frac{13}{5}}{x+4}+\frac{\frac{2}{5}}{x-1} &= \frac{\frac{13}{5}(x-1)}{(x+1)(x-4)}+\frac{\frac{2}{5}(x+4)}{(x-4)(x+1)}=\frac{\frac{13}{5}(x-1)+\frac{2}{5}(x+4)}{(x+1)(x-4)} \\ &= \frac{\frac{13}{5}x-\frac{13}{5}+\frac{2}{5}x+\frac{8}{5}}{(x+1)(x-4)}=\frac{\frac{15}{5}x-\frac{5}{5}}{x^2+3x-4}=\frac{3x-1}{x^2+3x-4} \end{aligned}$$

Now we can easily integrate:

$$\begin{aligned} \int\frac{3x-1}{x^2+3x-4}\,dx &= \int\frac{\frac{13}{5}}{x+4}+\frac{\frac{2}{5}}{x-1}\,dx=\int\frac{\frac{13}{5}}{x+4}\,dx+\int\frac{\frac{2}{5}}{x-1}\,dx \\ &= \frac{13}{5}\int\frac{1}{x+4}\,dx+\frac{2}{5}\int\frac{1}{x-1}\,dx=\frac{13}{5}\ln|x+4|+\frac{2}{5}\ln|x-1|+C \end{aligned}$$

Thus the final answer is

$$\int\frac{x^4+x^3-5x^2+26x-21}{x^2+3x-4}\,dx=$$

$$\begin{aligned} &= \frac{x^3}{3}-x^2+5x+C_1+\int\frac{3x-1}{x^2+3x-4}\,dx=\frac{x^3}{3}-x^2+5x+C_1+\frac{13}{5}\ln|x+4|+\frac{2}{5}\ln|x-1|+C_2 \\ &= \boxed{\frac{x^3}{3}-x^2+5x+\frac{13}{5}\ln|x+4|+\frac{2}{5}\ln|x-1|+C} \end{aligned}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+4}+\frac{B}{x-1}=\frac{3x-1}{x^2+3x-4}$$

We bring the fractions to the common denominator:

$$\frac{A\left(x-1\right)}{\left(x+4\right)\left(x-1\right)}+\frac{B\left(x+4\right)}{\left(x+4\right)\left(x-1\right)}=\frac{3x-1}{\left(x+1\right)\left(x-4\right)}$$

and then multiply both sides by the denominator:

$$A\left(x-1\right)+B\left(x+4\right)=3x-1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=1$ into both sides:

$$\begin{aligned}A\left(1-1\right)+B\left(1+4\right)&=3x-1\\A\cdot 0+B\cdot 5&=3\cdot 1-1\\5B&=2\\B&=\frac{2}{5}\end{aligned}$$

Let us substitute $x=-4$ into both sides:

$$\begin{aligned}A\left(-4-1\right)+B\left(-4+4\right)&=3\left(-4\right)-1\\-5A&=-13\\A&=\frac{13}{5}\end{aligned}$$

and so $A=\dfrac{4}{5}$ and $B=\dfrac{11}{5}$.

5. $\displaystyle\int\frac{x^2+x-3}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\,dx$

Solution: We re-write the fraction $\dfrac{x^2+x-3}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}$ as a sum (or difference) of fractions with denominators $x+1$, $x-2$ and $x-5$. This means that we need to solve for A, B, and C in the equation

$$\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-5}=\frac{x^2+x-3}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned}\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-5}&=\frac{A\left(x-2\right)\left(x-5\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}+\frac{B\left(x+1\right)\left(x-5\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}+\frac{C\left(x+1\right)\left(x-2\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\\&=\frac{A\left(x^2-7x+10\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}+\frac{B\left(x^2-4x-5\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}+\frac{C\left(x^2-x-2\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\\&=\frac{A\left(x^2-7x+10\right)+B\left(x^2-4x-5\right)+C\left(x^2-x-2\right)}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\\&=\frac{Ax^2-7Ax+10A+Bx^2-4Bx-5B+Cx^2-Cx-2C}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\\&=\frac{\left(A+B+C\right)x^2+\left(-7A-4B-C\right)x+10A-5B-2C}{\left(x+1\right)\left(x-2\right)\left(x-5\right)}\end{aligned}$$

Thus
$$\frac{(A+B+C)\,x^2+(-7A-4B-C)\,x+10A-5B-2C}{(x+1)\,(x-2)\,(x-5)}=\frac{x^2+x-3}{(x+1)\,(x-2)\,(x-5)}$$
We clear the denominators by multiplication
$$(A+B+C)\,x^2+(-7A-4B-C)\,x+10A-5B-2C=x^2+x-3$$
The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. We have an equation for each coefficient that gives us a system of linear equations:
$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ 10A & - & 5B & - & 2C & = & -3 \end{array}$$
We solve the system by elimination: first we will eliminate C from the second and third equations. To eliminate C from the second equation, we simply add the first and second equations.
$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ \hline -6A & - & 3B & & & = & 2 \end{array}$$
To eliminate C from the third equation, we multiply the first equation by 2 and add that to the third equation.
$$\begin{array}{rcrcrcr} 2A & + & 2B & + & 2C & = & 2 \\ 10A & - & 5B & - & 2C & = & -3 \\ \hline 12A & - & 3B & & & = & -1 \end{array}$$
We now have a system of linear equations in two variables:
$$\begin{aligned} -6A-3B &= 2 \\ 12A-3B &= -1 \end{aligned}$$
We will eliminate B by adding the opposite of the first equation to the second equation.
$$\begin{array}{rcrcr} 6A & + & 3B & = & -2 \\ 12A & - & 3B & = & -1 \\ \hline & & 18A & = & -3 \\ & & A & = & -\dfrac{1}{6} \end{array}$$
Using the equation $6A+3B=-2$ we can now solve for B.
$$\begin{aligned} 6\left(-\frac{1}{6}\right)+3B &= -2 \\ -1+3B &= -2 \\ 3B &= -1 \\ B &= -\frac{1}{3} \end{aligned}$$
Using the first equation, we can now solve for C.
$$\begin{aligned} A+B+C &= 1 \\ -\frac{1}{6}+\left(-\frac{1}{3}\right)+C &= 1 \\ -\frac{1}{2}+C &= 1 \\ C &= \frac{3}{2} \end{aligned}$$

Thus $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{3}{2}$

So we have that our fraction, $\frac{x^2+x-3}{(x+1)(x-2)(x-5)}$ can be re-written as $\frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5}$. We check:

$$\begin{aligned}
\frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5} &= \frac{-\frac{1}{6}(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{-\frac{1}{3}(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{\frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}(x-2)(x-5) - \frac{1}{3}(x+1)(x-5) + \frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}\left(x^2-7x+10\right) - \frac{1}{3}\left(x^2-4x-5\right) + \frac{3}{2}\left(x^2-x-2\right)}{(x+1)(x-2)(x-5)} \\
&= \frac{-\frac{1}{6}x^2 + \frac{7}{6}x - \frac{5}{3} - \frac{1}{3}x^2 + \frac{4}{3}x + \frac{5}{3} + \frac{3}{2}x^2 - \frac{3}{2}x - 3}{(x+1)(x-2)(x-5)} \\
&= \frac{\left(-\frac{1}{6} - \frac{1}{3} + \frac{3}{2}\right)x^2 + \left(\frac{7}{6} + \frac{4}{3} - \frac{3}{2}\right)x - \frac{5}{3} + \frac{5}{3} - 3}{(x+1)(x-2)(x-5)} \\
&= \frac{x^2+x-3}{(x+1)(x-2)(x-5)}
\end{aligned}$$

Now we can easily integrate:

$$\begin{aligned}
\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx &= \int \frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5}\,dx \\
&= -\frac{1}{6}\int \frac{1}{x+1}\,dx - \frac{1}{3}\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-5}\,dx \\
&= \boxed{-\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{2}{3}\ln|x-5| + C}
\end{aligned}$$

Method 2: The values of A, B, and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-5} = \frac{x^2+x-3}{(x+1)(x-2)(x-5)}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{B(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x-5)} = \frac{x^2+x-3}{(x+1)(x-2)(x-5)}$$

and then multiply both sides by the denominator:

$$A(x-2)(x-5) + B(x+1)(x-5) + C(x+1)(x-2) = x^2+x-3$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x = 2$ into both sides:

$$\begin{aligned}
A(2-2)(2-5) + B(2+1)(2-5) + C(2+1)(2-2) &= 2^2+2-3 \\
0A - 9B + 0C &= 3 \\
-9B &= 3 \\
B &= -\frac{1}{3}
\end{aligned}$$

Let us substitute $x = -1$ into both sides:

$$\begin{aligned} A(-1-2)(-1-5) + B(-1+1)(-1-5) + C(-1+1)(-1-2) &= (-1)^2 + (-1) - 3 \\ A(-3)(-6) + 0B + 0C &= -3 \\ 18A &= -3 \\ A &= -\frac{1}{6} \end{aligned}$$

Let us substitute $x = 5$ into both sides:

$$\begin{aligned} A(5-2)(5-5) + B(5+1)(5-5) + C(5+1)(5-2) &= 5^2 + 5 - 3 \\ A(0) + B(0) + C(6)(3) &= 27 \\ 18C &= 27 \\ C &= \frac{3}{2} \end{aligned}$$

and so $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$.

6. $\displaystyle\int \frac{2x-1}{(x-5)^2}\,dx$

Solution: We will re-write the fraction $\frac{2x-1}{(x-5)^2}$ as a sum (or difference) of fractions with denominators $x-5$ and $(x-5)^2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{A(x-5)+B}{(x-5)^2} = \frac{Ax-5A+B}{(x-5)^2}$$

Thus we have

$$\frac{Ax-5A+B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We clear the denominators by multiplication

$$Ax - 5A + B = 2x - 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A &= 2 \\ -5A + B &= -1 \end{aligned}$$

We solve this system and obtain $A = 2$ and $B = 9$.

So our fraction, $\frac{2x-1}{(x-5)^2}$ can be re-written as $\frac{2}{x-5} + \frac{9}{(x-5)^2}$. We check:

$$\frac{2}{x-5} + \frac{9}{(x-5)^2} = \frac{2(x-5)}{(x-5)^2} + \frac{9}{(x-5)^2} = \frac{2x-10+9}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

Now we can easily integrate:

$$\int \frac{2x-1}{(x-5)^2}\,dx = \int \frac{2}{x-5} + \frac{9}{(x-5)^2}\,dx = 2\int \frac{1}{x-5}\,dx + 9\int \frac{1}{(x-5)^2}\,dx = \boxed{2\ln|x-5| - \frac{9}{x-5} + C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We bring the fractions to the common denominator:

$$\frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

and then multiply both sides by the denominator:

$$A(x-5) + B = 2x-1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x = 5$ into both sides:

$$\begin{aligned} A(0) + B &= 9 \\ B &= 9 \end{aligned}$$

The other value of x can be arbitrarily chosen. (There is no value that would eliminate B from the equation.) For easy substitution, let us substitute $x = 0$ into both sides and also substitute $B = 9$:

$$\begin{aligned} A(-5) + 9 &= -1 \\ -5A &= -10 \\ A &= 2 \end{aligned}$$

and so $A = 2$ and $B = 9$.

7. $\displaystyle\int \frac{x+3}{(x-1)^3}\,dx$

Solution: We re-write the fraction $\dfrac{x+3}{(x-1)^3}$ as a sum (or difference) of fractions with denominators $x-1$, $(x-1)^2$ and $(x-1)^3$. This means that we need to solve for A, B, and C in the equation

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} &= \frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} \\ &= \frac{A\left(x^2-2x+1\right) + B(x-1) + C}{(x-1)^3} = \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3} \\ &= \frac{Ax^2 + (-2A+B)x + A - B + C}{(x-1)^3} \end{aligned}$$

Thus

$$\frac{Ax^2+(-2A+B)\,x+A-B+C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

We clear the denominators by multiplication

$$Ax^2+(-2A+B)\,x+A-B+C=x+3$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. We have an equation for each coefficient that gives us a system of linear equations:

$$\begin{array}{rcrcrcl} A & & & & & = & 0 \\ -2A & + & B & & & = & 1 \\ A & - & B & + & C & = & 3 \end{array}$$

Since $A=0$, this is really a system in two variables:

$$\begin{aligned} B &= 1 \\ -B+C &= 3 \end{aligned}$$

We solve this system and obtain $B=1$ and $C=4$.

So our fraction, $\dfrac{x+3}{(x-1)^3}$ can be re-written as $\dfrac{1}{(x-1)^2}+\dfrac{4}{(x-1)^3}$. We check:

$$\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}=\frac{1\,(x-1)}{(x-1)^3}+\frac{4}{(x-1)^3}=\frac{x-1+4}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

Now we can easily integrate:

$$\begin{aligned} \int\frac{x+3}{(x-1)^3}\,dx &= \int\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}\,dx=\int\frac{1}{(x-1)^2}\,dx+4\int\frac{1}{(x-1)^3}\,dx \\ &= -\frac{1}{x-1}-\frac{4}{2}\cdot\frac{1}{(x-1)^2}+C=\boxed{-\frac{1}{x-1}-\frac{2}{(x-1)^2}+C} \\ &= \frac{-1\,(x-1)}{(x-1)^2}-\frac{2}{(x-1)^2}+C=\frac{-x+1-2}{(x-1)^2}+C=\boxed{\frac{-x-1}{(x-1)^2}+C} \end{aligned}$$

Both final answers are acceptable.

Method 2: The values of A, B, and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

We bring the fractions to the common denominator:

$$\frac{A\,(x-1)^2}{(x-1)^3}+\frac{B\,(x-1)}{(x-1)^3}+\frac{C}{(x-1)^3}=\frac{x+3}{(x-1)^3}$$

and then multiply both sides by the denominator:

$$A\,(x-1)^2+B\,(x-1)+C=x+3$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=1$ into both sides:

$$\begin{aligned} A\,(0)+B\,(0)+C &= 1+3 \\ C &= 4 \end{aligned}$$

There is no value other than 1 that would eliminate A or B from the equation. Our method will still work. For easy substitution, let us substitute $x = 0$ into both sides and also substitute $C = 4$:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(0-1)^2 + B(0-1) + 4 &= 0+3 \\ A - B + 4 &= 3 \\ A - B &= -1 \end{aligned}$$

Let us substitute $x = 2$ into both sides:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(2-1)^2 + B(2-1) + 4 &= 2+3 \\ A + B + 4 &= 5 \\ A + B &= 1 \end{aligned}$$

We now solve the system of equations

$$\begin{aligned} A - B &= -1 \\ A + B &= 1 \end{aligned}$$

and obtain $A = 0$ and $B = 1$. Recall that we already have $C = 4$.

8. $\displaystyle\int \frac{x^4}{x^4-1}\,dx$

Solution: This rational function is an improper fraction since the numerator has the same degree as the denominator. We first perform long division. This one is an easy one; the method featured below is called smuggling.

$$\frac{x^4}{x^4-1} = \frac{x^4-1+1}{x^4-1} = \frac{x^4-1}{x^4-1} + \frac{1}{x^4-1} = 1 + \frac{1}{x^4-1}$$

Thus

$$\int \frac{x^4}{x^4-1}\,dx = \int 1 + \frac{1}{x^4-1}\,dx = \int 1\,dx + \int \frac{1}{x^4-1}\,dx = x + C_1 + \int \frac{1}{x^4-1}\,dx$$

We apply the method of partial fractions to compute $\displaystyle\int \frac{1}{x^4-1}\,dx$.

We factor the denominator: $x^4 - 1 = \left(x^2+1\right)(x+1)(x-1)$. Next, we re-write the fraction $\dfrac{1}{x^4-1}$ as a sum (or difference) of fractions with denominators x^2+1 and $x+1$, and $x-1$. In the fraction with quadratic denominator, the numerator is linear. This means that we need to solve for A and B in the equation

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} = \frac{1}{x^4-1}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} &= \frac{(Ax+B)(x+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{C\left(x^2+1\right)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{D\left(x^2+1\right)(x+1)}{(x^2+1)(x+1)(x-1)} \\ &= \frac{(Ax+B)(x+1)(x-1) + C\left(x^2+1\right)(x-1) + D\left(x^2+1\right)(x+1)}{x^4-1} \\ &= \frac{(Ax+B)\left(x^2-1\right) + C\left(x^3-x^2+x-1\right) + D\left(x^3+x^2+x+1\right)}{x^4-1} \\ &= \frac{Ax^3 + Bx^2 - Ax - B + Cx^3 - Cx^2 + Cx - C + Dx^3 + Dx^2 + Dx + D}{x^4-1} \\ &= \frac{(A+C+D)x^3 + (B-C+D)x^2 + (-A+C+D)x - B - C + D}{x^4-1} \end{aligned}$$

Thus
$$\frac{(A+C+D)\,x^3+(B-C+D)\,x^2+(-A+C+D)\,x-B-C+D}{x^4-1}=\frac{1}{x^4-1}$$
We clear the denominators by multiplication
$$(A+C+D)\,x^3+(B-C+D)\,x^2+(-A+C+D)\,x-B-C+D=1$$
The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:
$$\begin{array}{rrrrrrrrr} A & & & + & C & + & D & = & 0 \\ & & B & - & C & + & D & = & 0 \\ -A & & & + & C & + & D & = & 0 \\ & - & B & - & C & + & D & = & 1 \end{array}$$
We will solve this system by elimination. First, we will eliminate A using the first equation. The second and fourth equations do not have A in them, so there is nothing to do there. To eliminate A from the third equation, we add the first one to it.
$$\begin{array}{rrrrrrrr} & B & - & C & + & D & = & 0 \\ & & & 2C & + & 2D & = & 0 \\ - & B & - & C & + & D & = & 1 \end{array}$$
We now have three equations with three unknowns. We will use the first equation to eliminate B. In case of the second equation, again, there is nothing to do. We add the first equation to the third one to eliminate B.
$$\begin{array}{rrrrr} 2C & + & 2D & = & 0 \\ -2C & + & 2D & = & 1 \end{array}$$
Adding the two equations eliminates C and gives us $4D=1$ and so $D=\dfrac{1}{4}$. Next, we compute C using the equation
$$\begin{aligned} 2C+2D &= 0 \\ 2C+2\left(\frac{1}{4}\right) &= 0 \\ 2C &= -\frac{1}{2} \\ C &= -\frac{1}{4} \end{aligned}$$
We can compute A using the first equation, $A+C+D=0$
$$\begin{aligned} A+C+D &= 0 \\ A-\frac{1}{4}+\frac{1}{4} &= 0 \\ A &= 0 \end{aligned}$$
and we can compute B using the second equation,
$$\begin{aligned} B-C+D &= 0 \\ B-\left(-\frac{1}{4}\right)+\frac{1}{4} &= 0 \\ B &= -\frac{1}{2} \end{aligned}$$

Thus $A=0$, $B=-\frac{1}{2}$, $C=-\frac{1}{4}$, and $D=\frac{1}{4}$.

So our fraction, $\frac{1}{x^4-1}$ can be re-written as $\frac{-\frac{1}{2}}{x^2+1}-\frac{\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}$. We check:

$$\begin{aligned}
\frac{-\frac{1}{2}}{x^2+1}-\frac{\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1} &= \frac{-\frac{1}{2}(x+1)(x-1)}{(x^2+1)(x+1)(x-1)}-\frac{\frac{1}{4}\left(x^2+1\right)(x-1)}{(x^2+1)(x+1)(x-1)}+\frac{\frac{1}{4}\left(x^2+1\right)(x+1)}{(x^2+1)(x+1)(x-1)} \\
&= \frac{-\frac{1}{2}(x+1)(x-1)-\frac{1}{4}\left(x^2+1\right)(x-1)+\frac{1}{4}\left(x^2+1\right)(x+1)}{(x^2+1)(x+1)(x-1)} \\
&= \frac{-\frac{1}{2}\left(x^2-1\right)-\frac{1}{4}\left(x^3-x^2+x-1\right)+\frac{1}{4}\left(x^3+x^2+x+1\right)}{x^4-1} \\
& \quad \frac{-\frac{1}{2}x^2+\frac{1}{2}-\frac{1}{4}x^3+\frac{1}{4}x^2-\frac{1}{4}x+\frac{1}{4}+\frac{1}{4}x^3+\frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{4}}{x^4-1} \\
&= \frac{\left(-\frac{1}{4}+\frac{1}{4}\right)x^3+\left(-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)x^2+\left(-\frac{1}{4}+\frac{1}{4}\right)x+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}}{x^4-1}=\frac{1}{x^4-1}
\end{aligned}$$

Now we re-write the integral:

$$\begin{aligned}
\int \frac{1}{x^4-1}\,dx &= \int \frac{-\frac{1}{2}}{x^2+1}-\frac{\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}\,dx = -\frac{1}{2}\int \frac{1}{x^2+1}\,dx-\frac{1}{4}\int \frac{1}{x+1}\,dx+\frac{1}{4}\int \frac{1}{x-1}\,dx \\
&= -\frac{1}{2}\arctan x-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|+C
\end{aligned}$$

Thus the final answer is

$$\begin{aligned}
\int \frac{x^4}{x^4-1}\,dx &= \int 1+\frac{x^4}{x^4-1}\,dx = \int 1\,dx+\int \frac{1}{x^4-1}\,dx \\
&= x+C_1-\frac{1}{2}\arctan x-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|+C_2 \\
&= \boxed{x-\frac{1}{2}\arctan x-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|+C}
\end{aligned}$$

Method 2: The values of A, B, C, and D can be found using a slightly different method as follows. Consider first the equation

$$\frac{Ax+B}{x^2+1}+\frac{C}{x+1}+\frac{D}{x-1}=\frac{1}{x^4-1}$$

We bring the fractions to the common denominator:

$$\frac{(Ax+B)(x+1)(x-1)}{(x^2+1)(x+1)(x-1)}+\frac{C\left(x^2+1\right)(x-1)}{(x^2+1)(x+1)(x-1)}+\frac{D\left(x^2+1\right)(x+1)}{(x^2+1)(x+1)(x-1)}=\frac{1}{(x^2+1)(x+1)(x-1)}$$

and then multiply both sides by the denominator:

$$(Ax+B)(x+1)(x-1)+C\left(x^2+1\right)(x-1)+D\left(x^2+1\right)(x+1)=1$$

The equation above is about two functions; the two sides must be equal for all values of x. Let us substitute $x=-1$ into both sides:

$$\begin{aligned}
(Ax+B)(x+1)(x-1)+C\left(x^2+1\right)(x-1)+D\left(x^2+1\right)(x+1) &= 1\\
(A(-1)+B)((-1)+1)((-1)-1)+C\left((-1)^2+1\right)((-1)-1)+D\left((-1)^2+1\right)((-1)+1) &= 1\\
(-A+B)(0)(-2)+C(2)(-2)+D(2)(0) &= 1\\
-4C &= 1\\
C &= -\frac{1}{4}
\end{aligned}$$

Let us substitute $x=1$ into both sides:

$$\begin{aligned}
(Ax+B)(x+1)(x-1)+C\left(x^2+1\right)(x-1)+D\left(x^2+1\right)(x+1) &= 1\\
(A(1)+B)((1)+1)(1-1)+C\left(1^2+1\right)(1-1)+D\left(1^2+1\right)(1+1) &= 1\\
(A+B)(2)(0)+C(2)(0)+D(2)(2) &= 1\\
4D &= 1\\
D &= \frac{1}{4}
\end{aligned}$$

Let us substitute $x=0$ into both sides and also $C=-\frac{1}{4}$ and $D=\frac{1}{4}$:

$$\begin{aligned}
(Ax+B)(x+1)(x-1)+C\left(x^2+1\right)(x-1)+D\left(x^2+1\right)(x+1) &= 1\\
(A(0)+B)((0)+1)(0-1)+C\left(0^2+1\right)(0-1)+D\left(0^2+1\right)(0+1) &= 1\\
(B)(1)(-1)+C(1)(-1)+D(1)(1) &= 1\\
-B-C+D &= 1\\
-B-\left(-\frac{1}{4}\right)+\frac{1}{4} &= 1\\
-B+\frac{1}{2} &= 1\\
-B &= \frac{1}{2}\\
B &= -\frac{1}{2}
\end{aligned}$$

Let us substitute $x=2$ into both sides and also $B=-\frac{1}{2}$, $C=-\frac{1}{4}$ and $D=\frac{1}{4}$:

$$\begin{aligned}
(Ax+B)(x+1)(x-1)+C\left(x^2+1\right)(x-1)+D\left(x^2+1\right)(x+1) &= 1\\
(A(2)+B)((2)+1)(2-1)+C\left(2^2+1\right)(2-1)+D\left(2^2+1\right)(2+1) &= 1\\
(2A+B)(3)(1)+C(5)(1)+D(5)(3) &= 1\\
3(2A+B)+5C+15D &= 1\\
6A+3B+5C+15D &= 1\\
6A+3\left(-\frac{1}{2}\right)+5\left(-\frac{1}{4}\right)+15\left(\frac{1}{4}\right) &= 1\\
6A-\frac{3}{2}-\frac{5}{4}+\frac{15}{4} &= 1\\
6A+\frac{-6-5+15}{4} &= 1
\end{aligned}$$

$$\begin{aligned} 6A + \frac{4}{4} &= 1 \\ 6A + 1 &= 1 \\ 6A &= 0 \\ A &= 0 \end{aligned}$$

and so $A = 0$, $B = -\dfrac{1}{2}$, $C = -\dfrac{1}{4}$, and $D = \dfrac{1}{4}$.

9. $\displaystyle\int \sec x \; dx = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C = \ln |\sec x + \tan x| + C = -\ln |\sec x - \tan x| + C$

Solution:

$$\int \sec x \; dx = \int \frac{1}{\cos x} \; dx = \int \frac{1}{\cos x} \cdot \frac{\cos x}{\cos x} \; dx = \int \frac{\cos x}{\cos^2 x} \; dx = \int \frac{\cos x}{1 - \sin^2 x} \; dx$$

Now let $u = \sin x$. Then $du = \cos x dx$.

$$\int \frac{\cos x}{1 - \sin^2 x} \; dx = \int \frac{1}{1 - \sin^2 x} \; \cos x dx = \int \frac{1}{1 - u^2} \; du = \int \frac{1}{(1 - u)(1 + u)} \; du$$

This integral can be computed by partial fractions:

$$\frac{A}{1 - u} + \frac{B}{1 + u} = \frac{1}{1 - u^2}$$

The left-hand side can be re-written

$$\frac{A}{1 - u} + \frac{B}{1 + u} = \frac{A(1 + u)}{(1 - u)(1 + u)} + \frac{B(1 - u)}{(1 + u)(1 - u)} = \frac{Au + A - Bu + B}{1 - u^2} = \frac{(A - B)u + A + B}{1 - u^2}$$

So we have

$$\frac{(A - B)u + A + B}{1 - u^2} = \frac{1}{1 - u^2}$$

We clear the denominators by multiplication

$$(A - B)u + A + B = 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficint by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{aligned} A - B &= 0 \\ A + B &= 1 \end{aligned}$$

we solve this system and obtain $A = B = \dfrac{1}{2}$. Indeed,

$$\frac{1}{2}\left(\frac{1}{1 + u} + \frac{1}{1 - u}\right) = \frac{1}{2}\frac{1 - u + 1 + u}{(1 - u)(1 + u)} = \frac{1}{2}\frac{2}{1 - u^2} = \frac{1}{1 - u^2}$$

Now for the integral:

$$\begin{aligned} \int \frac{1}{1 - u^2} \; du &= \int \frac{1}{(1 - u)(1 + u)} \; du = \int \frac{1}{2}\left(\frac{1}{1 - u} + \frac{1}{1 + u}\right) \; du = \frac{1}{2}\int \frac{1}{1 - u} \; du + \frac{1}{2}\int \frac{1}{1 + u} \; du \\ &= -\frac{1}{2}\ln|1 - u| + \frac{1}{2}\ln|1 + u| + C = \frac{1}{2}\ln|1 + u| - \frac{1}{2}\ln|1 - u| + C \\ &= \boxed{\frac{1}{2}\ln|1 + \sin x| - \frac{1}{2}\ln|1 - \sin x| + C} \end{aligned}$$

Note the $-$ sign in $\displaystyle\int \frac{1}{1-u}\,du = -\ln|1-u| + C$ is caused by the chain rule. Aslo note that the final answer can be re-written in several forms:

$$\frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| =$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|$$

$$= \ln\left|\left(\frac{(1+\sin x)^2}{\cos^2 x}\right)^{1/2}\right| = \ln\sqrt{\frac{(1+\sin x)^2}{\cos^2 x}} = \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| = \ln|\sec x + \tan x|$$

So, our result can also be presented as $\boxed{\ln|\sec x + \tan x| + C}$

Another form can be obtained as shown below.

$$\frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| =$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\cdot\frac{1-\sin x}{1-\sin x}\right| = \frac{1}{2}\ln\left|\frac{1-\sin^2 x}{(1-\sin x)^2}\right| = \frac{1}{2}\ln\left|\frac{\cos^2 x}{(1-\sin x)^2}\right|$$

$$= \ln\left|\left(\frac{\cos^2 x}{(1-\sin x)^2}\right)^{1/2}\right| = \ln\sqrt{\frac{\cos^2 x}{(1-\sin x)^2}} = \ln\left|\frac{\cos x}{1-\sin x}\right| = \ln\left|\frac{1}{\dfrac{1-\sin x}{\cos x}}\right|$$

$$= \ln\left|\left(\frac{1-\sin x}{\cos x}\right)^{-1}\right| = \ln\left|\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right|^{-1} = -\ln|\sec x - \tan x|$$

So, our result can also be presented as $\boxed{-\ln|\sec x - \tan x| + C}$. Actually, there are more forms possible, but we will stop here.

10. $\displaystyle\int \operatorname{csch} x\; dx$

Solution: This is an interesting application of partial fractions.

$$\int \operatorname{csch} x\; dx = \int \frac{2}{e^x - e^{-x}}\,dx = \int \frac{2}{e^x - \dfrac{1}{e^x}}\,dx$$

we now multiply both numerator and denominator by e^x.

$$\int \frac{2}{e^x - \dfrac{1}{e^x}}\,dx = \int \frac{2e^x}{(e^x)^2 - 1}\,dx$$

We proceed with a substitution: let $u = e^x$. Then $du = e^x dx$ and so

$$\int \frac{2}{(e^x)^2 - 1}\,(e^x dx) = \int \frac{2}{u^2-1}\,du$$

This is now an integral we can easily compute via partial fractions. We easily decompose $\dfrac{2}{u^2-1}$ as $\dfrac{1}{u-1}-\dfrac{1}{u+1}$

$$\begin{aligned}\int \operatorname{csch} x \; dx &= \int \frac{2}{u^2-1} du = \int \left(\frac{1}{u-1}-\frac{1}{u+1}\right) du = \int \frac{1}{u-1} du - \int \frac{1}{u+1} du = \ln|u-1| - \ln|u+1| + C \\ &= \boxed{\ln|e^x-1| - \ln(e^x+1) + C}\end{aligned}$$