Sample Problems

Compute each of the following integrals.

1. $\displaystyle\int xe^x\ dx$
2. $\displaystyle\int x\cos x\ dx$
3. $\displaystyle\int xe^{-4x}\ dx$
4. $\displaystyle\int \ln x\ dx$
5. $\displaystyle\int \sin^{-1} x\ dx$
6. $\displaystyle\int \tan^{-1} x\ dx$
7. $\displaystyle\int e^x\sin x\ dx$
8. $\displaystyle\int x^2\sin 5x\ dx$
9. $\displaystyle\int \sec^3 x\ dx$

Practice Problems

1. $\displaystyle\int xe^{2x}\ dx$
2. $\displaystyle\int xe^{-3x}\ dx$
3. $\displaystyle\int_0^{\ln 2} xe^{-3x}\ dx$
4. $\displaystyle\int \cos^{-1} x\ dx$
5. $\displaystyle\int x2^x\ dx$
6. $\displaystyle\int x^2 2^x\ dx$
7. $\displaystyle\int x\cos x\ dx$
8. $\displaystyle\int x^2\cos x\ dx$
9. $\displaystyle\int x\ln x\ dx$
10. $\displaystyle\int x^5\ln x\ dx$
11. $\displaystyle\int x\sin 10x\ dx$
12. $\displaystyle\int_1^9 \frac{\ln x}{\sqrt{x}}\ dx$
13. $\displaystyle\int e^x\sin 2x\ dx$
14. $\displaystyle\int_0^{\pi/4} x\sin 2x\ dx$
15. $\displaystyle\int \frac{x^3}{(x^2+2)^2}\ dx$
16. $\displaystyle\int \frac{\ln x}{x^7}\ dx$
17. $\displaystyle\int e^{5x}\cos 3x\ dx$

Sample Problems - Answers

1.) $xe^x - e^x + C$ 2.) $x\sin x + \cos x + C$ 3.) $-\frac{1}{16}e^{-4x} - \frac{1}{4}xe^{-4x} + C$ 4.) $x\ln x - x + C$

5.) $x\sin^{-1} x + \sqrt{1-x^2} + C$ 6.) $x\tan^{-1} x - \frac{1}{2}\ln\left(x^2+1\right) + C$ 7.) $\frac{1}{2}e^x(\sin x - \cos x) + C$

8.) $-\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C$ 9.) $\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

Practice Problems - Answers

1.) $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ 2.) $-\frac{1}{9}e^{-3x} - \frac{1}{3}xe^{-3x} + C$ 3.) $\frac{7}{72} - \frac{1}{24}\ln 2$ 4.) $x\cos^{-1} x - \sqrt{1-x^2} + C$

5.) $\frac{2^x}{\ln 2}\left(x - \frac{1}{\ln 2}\right) + C$ 6.) $\frac{2^x}{\ln 2}\left(x^2 - \frac{2x}{\ln 2} + \frac{2}{\ln^2 2}\right) + C$ 7.) $x\sin x + \cos x + C$

8.) $x^2\sin x + 2x\cos x - 2\sin x + C$ 9.) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$ 10.) $\frac{1}{6}x^6\ln x - \frac{1}{36}x^6 + C$

11.) $\frac{1}{100}\sin 10x - \frac{1}{10}x\cos 10x + C$ 12.) $6\ln 9 - 8$ 13.) $\frac{1}{5}e^x\sin 2x - \frac{2}{5}e^x\cos 2x + C$ 14) $\frac{1}{4}$

15.) $\frac{1}{2}\ln\left(x^2+2\right) + \frac{1}{x^2+2} + C$ 16.) $-\frac{1}{36x^6} - \frac{1}{6x^6}\ln x + C$ 17.) $\frac{5}{34}(\cos 3x)e^{5x} + \frac{3}{34}(\sin 3x)e^{5x} + C$

Sample Problems - Solutions

1. $\displaystyle\int xe^x\ dx$

Solution: We will integrate this by parts, using the formula

$$\int u\ dv = uv - \int v\ du$$

Let $u = x$ and $dv = e^x dx$ Then we obtain du and v by differentiation and integration

$$\frac{du}{dx} = 1 \text{ and so } du = dx \text{ and } v = \int dv = \int e^x dx = e^x + C \qquad \text{we wil use } C = 0$$

We summarize these results in the table below:

$v = e^x$	$u = x$
$dv = e^x dx$	$du = dx$

$$\begin{aligned} \int u\ dv &= uv - \int v\ du \quad \text{becomes} \\ \int xe^x\ dx &= xe^x - \int e^x dx = \boxed{xe^x - e^x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x$$

2. $\displaystyle\int x\cos x\ dx$

Solution: Let $u = x$ and $dv = \cos x\ dx$ Then we obtain du and v by differentiation and integration. $v = \displaystyle\int dv = \int \cos x\ dx = \sin x + C$ (we will use $C = 0$) and $\dfrac{du}{dx} = 1 \implies du = dx$. We summarize these results in the table below:

$v = \sin x$	$u = x$
$dv = \cos x dx$	$du = 1dx$

$$\begin{aligned} \int u\ dv &= uv - \int v\ du \quad \text{becomes} \\ \int x\cos x\ dx &= x\sin x - \int \sin x\ dx = x\sin x - (-\cos x) = \boxed{x\sin x + \cos x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$\frac{d}{dx}\left(x\sin x + \cos x + C\right) = \sin x + x\cos x - \sin x = x\cos x$$

3. $\displaystyle\int xe^{-4x}\,dx$

Solution: During the computation, we will see some sort of a "good news-bad news" situation. The bad news is that in the course of the computation, we will run into two integrals that require substitution. The good news is that we only have to compute once because the two integrands are identical. This will happen quite often when integrating by parts, especially when exponential equations are involved.

Let $u = x$ and $dv = e^{-4x}dx$ Then we obtain du and v by differentiation and integration.

$$u = x \implies \frac{du}{dx} = 1 \implies du = 1dx \quad \text{and} \quad v = \int dv = \int e^{-4x}\,dx$$

To compute v, we will integrate by substitution. Let $w = -4x$ then $\dfrac{dw}{dx} = -4$ and so $dx = \dfrac{dw}{-4}$

$$\int e^{-4x}\,dx = \int e^{w}\,\frac{dw}{-4} = -\frac{1}{4}\int e^{w}\,dw = -\frac{1}{4}e^{w} + C = -\frac{1}{4}e^{-4x} + C$$

We will choose $C = 0$ and so $v = -\dfrac{1}{4}e^{-4x}$. We summarize all this in the table below:

$v = -\frac{1}{4}e^{-4x}$	$u = x$
$dv = e^{-4x}dx$	$du = dx$

$$\begin{aligned}
\int u\,dv &= uv - \int v\,du \quad \text{becomes} \\
\int xe^{-4x}\,dx &= -\frac{1}{4}xe^{-4x} - \int -\frac{1}{4}e^{-4x}\,dx = -\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}\,dx = -\frac{1}{4}xe^{-4x} + \frac{1}{4}\left(-\frac{1}{4}e^{-4x}\right) + C \\
&= \boxed{-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C}
\end{aligned}$$

We check our result by differentiating the answer.

$$\frac{d}{dx}\left(-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C\right) =$$

$$\begin{aligned}
&= -\frac{1}{4}\left(\frac{d}{dx}\left(xe^{-4x}\right)\right) - \frac{1}{16}\frac{d}{dx}e^{-4x} = -\frac{1}{4}\left(e^{-4x} + x\left(-4e^{-4x}\right)\right) - \frac{1}{16}\left(-4e^{-4x}\right) \\
&= -\frac{1}{4}e^{-4x} + xe^{-4x} + \frac{1}{4}e^{-4x} = xe^{-4x}
\end{aligned}$$

4. $\displaystyle\int \ln x\,dx$

Solution: Let $u = \ln x$ and $dv = 1dx$ Then we obtain du and v by differentiation and integration.

$$u = \ln x \implies \frac{du}{dx} = \frac{1}{x} \implies du = \frac{1}{x}dx \quad \text{and} \quad v = \int dv = \int 1dx = x$$

We summarize all this in the table below:

$v = x$	$u = \ln x$
$dv = 1dx$	$du = \frac{1}{x}dx$

$$\begin{aligned}
\int u\,dv &= uv - \int v\,du \quad \text{becomes} \\
\int \ln x\,dx &= x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int 1\,dx = \boxed{x\ln x - x + C}
\end{aligned}$$

We check our result by differentiating the answer.

$$\frac{d}{dx}\left(x\ln x - x + C\right) = \ln x + x\cdot\frac{1}{x} - 1 = \ln x$$

5. $\displaystyle\int \sin^{-1} x\ dx$

Solution: Let $u = \sin^{-1} x$ and $dv = 1dx$. We obtain du and v by differentiation and integration.

$$u = \sin^{-1} x \quad\Longrightarrow\quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad\Longrightarrow\quad du = \frac{1}{\sqrt{1-x^2}}dx \quad\text{and}\quad v = \int dv = \int 1dx = x$$

We summarize all this in the table below:

$v = x$	$u = \sin^{-1} x$
$dv = 1dx$	$du = \dfrac{1}{\sqrt{1-x^2}}dx$

$$\begin{aligned}\int u\ dv &= uv - \int v\ du \quad \text{becomes}\\ \int \sin^{-1} x\ dx &= x\sin^{-1} x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\ dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\ dx\end{aligned}$$

We compute the integral $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\ dx$ by substitution. Let $w = 1-x^2$. Then $\dfrac{dw}{dx} = -2x$ and so $dx = \dfrac{dw}{-2x}$.

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\ dx &= \int \frac{x}{\sqrt{1-x^2}}\ dx = \int \frac{x}{\sqrt{w}}\ \frac{dw}{-2x} = -\frac{1}{2}\int \frac{1}{\sqrt{w}}\ dw = -\frac{1}{2}\int w^{-1/2}\ dw\\ &= -\frac{1}{2}\frac{w^{1/2}}{\frac{1}{2}} + C = -\sqrt{w} + C = -\sqrt{1-x^2} + C\end{aligned}$$

Thus the entire integral is

$$\int \sin^{-1} x\ dx = x\sin^{-1} x - \left(-\sqrt{1-x^2}\right) + C = \boxed{x\sin^{-1} x + \sqrt{1-x^2} + C}$$

We check our result by differentiating the answer.

$$\frac{d}{dx}\left(x\sin^{-1} x + \sqrt{1-x^2} + C\right) =$$

$$\begin{aligned}&= \frac{d}{dx}\left(x\sin^{-1} x\right) + \frac{d}{dx}\left(1-x^2\right)^{1/2} = \sin^{-1} x + x\cdot\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\\ &= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x\end{aligned}$$

6. $\displaystyle\int \tan^{-1} x\ dx$

Solution: Let $u = \tan^{-1} x$ and $dv = 1dx$. Then we obtain du and v by differentiation and integration.

$$u = \tan^{-1} x \quad\Longrightarrow\quad \frac{du}{dx} = \frac{1}{x^2+1} \quad\Longrightarrow\quad du = \frac{1}{x^2+1}dx \quad\text{and}\quad v = \int dv = \int 1dx = x$$

We summarize all this in the table below:

$v = x$	$u = \tan^{-1} x$
$dv = 1dx$	$du = \dfrac{1}{x^2+1}dx$

$$\int u\ dv = uv - \int v\ du \text{ becomes}$$
$$\int \tan^{-1} x\ dx = x\tan^{-1} x - \int x \cdot \frac{1}{x^2+1}\ dx = x\tan^{-1} x - \int \frac{x}{x^2+1}\ dx$$

We compute the integral $\int \frac{x}{x^2+1}\ dx$ by substitution. Let $w = x^2+1$. Then $dw = 2xdx$. We will not solve for dx, instead, we will take a bit of a shortcut.

$$\int \frac{x}{x^2+1}\ dx = \int \frac{\left(\frac{1}{2}\right)(2)\,x}{x^2+1}\ dx = \frac{1}{2}\int \frac{1}{x^2+1}\ (2xdx) = \frac{1}{2}\int \frac{1}{w}\ dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$$

Notice that we did not need the absolute value sign because x^2+1 is always positive. Now the entire integral is

$$\int \tan^{-1} x\ dx = \boxed{x\tan^{-1} x - \frac{1}{2}\ln\left(x^2+1\right) + C}$$

We check our result by differentiating the answer.

$$\frac{d}{dx}\left(x\tan^{-1} x - \frac{1}{2}\ln\left(x^2+1\right) + C\right) =$$

$$= \frac{d}{dx}\left(x\tan^{-1} x\right) - \frac{1}{2}\frac{d}{dx}\ln\left(x^2+1\right) = 1\cdot\arctan x + x\cdot\frac{1}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}(2x)$$
$$= \tan^{-1} x + \frac{x}{x^2+1} - \frac{x}{x^2+1} = \tan^{-1} x$$

7. $\int e^x \sin x\ dx$

Solution: This is an interesting application of integration by parts. Let M denote the integral $\int e^x \sin x\ dx$. Let $u = \sin x$ and $dv = e^x dx$ Then we obtain du and v by differentiation and integration.

$$u = \sin x \implies \frac{du}{dx} = \cos x \implies du = \cos x dx \quad \text{and} \quad v = \int dv = \int e^x dx = e^x$$

We summarize all this in the table below:

$v = e^x$	$u = \sin x$
$dv = e^x dx$	$du = \cos x dx$

$$\int u\ dv = uv - \int v\ du \quad \text{becomes}$$
$$\int (\sin x)\left(e^x\right)\ dx = (\sin x)\left(e^x\right) - \int e^x \cos x\ dx = e^x \sin x - \int e^x \cos x\ dx$$

$$\text{Thus} \quad \int e^x \sin x\ dx = e^x \sin x - \int e^x \cos x\ dx$$

It looks like our method produced a new integral, $\int e^x \cos x \, dx$ that also requires integration by parts. We proceed: let $u = \cos x$ and $dv = e^x dx$ Then we obtain du and v by differentiation and integration.

$$u = \cos x \implies \frac{du}{dx} = -\sin x \implies du = -\sin x dx \quad \text{and} \quad v = \int dv = \int e^x dx = e^x$$

We summarize all this in the table below:

$v = e^x$	$u = \cos x$
$dv = e^x dx$	$du = -\sin x dx$

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \quad \text{becomes} \\ \int (\cos x)(e^x) \, dx &= (\cos x)(e^x) - \int e^x (-\sin x) \, dx = e^x \cos x + \int e^x \sin x \, dx \end{aligned}$$

$$\text{Thus} \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now we obtained the original integral, $\int e^x \sin x$. At this point, it looks like we are getting nowhere because we are going in circles. However, this is not the case. Recall that we denote $\int e^x \sin x$ by M. Let us review the computation again:

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx \\ \int e^x \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

This is the same as

$$M = e^x \sin x - e^x \cos x - M$$

This is an equation that we can solve for M.

$$\begin{aligned} 2M &= e^x \sin x - e^x \cos x \\ M &= \frac{1}{2} e^x (\sin x - \cos x) \end{aligned}$$

Thus the answer is $\boxed{\frac{1}{2} e^x (\sin x - \cos x) + C}$. We check our result by differentiation.

$$\frac{d}{dx}\left(\frac{1}{2} e^x (\sin x - \cos x) \right) =$$

$$\begin{aligned} &= \frac{1}{2}\left(\frac{d}{dx} e^x \right)(\sin x - \cos x) + \frac{1}{2} e^x \frac{d}{dx}(\sin x - \cos x) = \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x (\cos x + \sin x) \\ &= \frac{1}{2} e^x (\sin x - \cos x + \sin x + \cos x) = \frac{1}{2} e^x (2 \sin x) = e^x \sin x \end{aligned}$$

8. $\displaystyle\int x^2 \sin 5x \, dx$

Solution: We will need to integrate by parts twice. First, let $u = x^2$ and $dv = \sin 5x dx$. Then we obtain du and v by differentiation and integration.

$$u = x^2 \implies \frac{du}{dx} = 2x \implies du = 2x dx \quad \text{and} \quad v = \int dv = \int \sin 5x \, dx = -\frac{1}{5}\cos 5x$$

We summarize all this in the table below:

$v = -\frac{1}{5}\cos 5x$	$u = x^2$
$dv = \sin 5x dx$	$du = 2x dx$

$$\int u \, dv = uv - \int v \, du \quad \text{becomes}$$

$$\int x^2 \sin 5x \, dx = \left(x^2\right)\left(-\frac{1}{5}\cos 5x\right) - \int \left(-\frac{1}{5}\cos 5x\right) 2x dx = -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\int x\cos 5x \, dx$$

We compute the integral $\displaystyle\int x \cos 5x \, dx$ by parts. Let $u = x$ and $dv = \cos 5x dx$. We obtain du and v by differentiation and integration.

$$u = x \implies du = dx \quad \text{and} \quad v = \int dv = \int \cos 5x \, dx = \frac{1}{5}\sin 5x$$

We summarize all this in the table below:

$v = \frac{1}{5}\sin 5x$	$u = x$
$dv = \cos 5x dx$	$du = dx$

$$\int u \, dv = uv - \int v \, du \quad \text{becomes}$$

$$\begin{aligned}\int x\cos 5x \, dx &= (x)\left(\frac{1}{5}\sin 5x\right) - \int \frac{1}{5}\sin 5x \, dx = \frac{1}{5}x\sin 5x - \frac{1}{5}\int \sin 5x \, dx \\ &= \frac{1}{5}x\sin 5x - \frac{1}{5}\cdot\frac{1}{5}\left(-\cos 5x\right) + C = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C\end{aligned}$$

Now the entire integral is

$$\begin{aligned}\int x^2\sin 5x \, dx &= -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\int x\cos 5x \, dx = -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\left(\frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C\right) \\ &= \boxed{-\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C}\end{aligned}$$

We check our result by differentiating the answer.

$$\frac{d}{dx}\left(-\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C\right) =$$

$$\begin{aligned} &= -\frac{1}{5}\frac{d}{dx}\left(x^2\cos 5x\right) + \frac{2}{25}\frac{d}{dx}\left(x\sin 5x\right) + \frac{2}{125}\frac{d}{dx}\cos 5x = \\ &= -\frac{1}{5}\left(2x\cos 5x + x^2\left(-5\sin 5x\right)\right) + \frac{2}{25}\left(1\cdot\sin 5x + x\left(5\cos 5x\right)\right) + \frac{2}{125}\left(5\left(-\sin 5x\right)\right) \\ &= -\frac{1}{5}\left(2x\cos 5x - 5x^2\sin 5x\right) + \frac{2}{25}\left(\sin 5x + 5x\cos 5x\right) + \frac{2}{125}\left(-5\sin 5x\right) \\ &= -\frac{2}{5}x\cos 5x + x^2\sin 5x + \frac{2}{25}\sin 5x + \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x = x^2\sin 5x\end{aligned}$$

9. $\displaystyle\int \sec^3 x \, dx =$

Solution: Let $u = \sec x$ and $dv = \sec^2 x dx$. Then we obtain du and v by differentiation and integration.

$$u = \sec x \implies \frac{du}{dx} = \tan x \sec x \implies du = \tan x \sec x dx \quad \text{and} \quad v = \int dv = \int \sec^2 x \, dx = \tan x$$

We summarize all this in the table below:

$v = \tan x$	$u = \sec x$
$dv = \sec^2 x dx$	$du = \tan x \sec x dx$

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \ \text{ becomes} \\ \int \sec x \sec^2 x \, dx &= \sec x \tan x - \int \tan x \tan x \sec x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx \end{aligned}$$

For the second integral, we will use that $\tan^2 x + 1 = \sec^2 x$.

$$\int \tan^2 x \sec x \, dx = \int \left(\sec^2 x - 1\right) \sec x \, dx = \int \sec^3 x - \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx$$

Recall that $\displaystyle\int \sec x \, dx = \ln |\sec x + \tan x| + C$. Thus we have that the second integral,

$$\int \tan^2 x \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx = \int \sec^3 x \, dx - \ln |\sec x + \tan x|$$

In summary, so far we have that

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx \\ &= \sec x \tan x - \left(\int \sec^3 x \, dx - \ln |\sec x + \tan x|\right) \\ &= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| \end{aligned}$$

We now have an equation in $\displaystyle\int \sec^3 x \, dx$ that we can easily solve.

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| \\ 2\int \sec^3 x \, dx &= \sec x \tan x + \ln |\sec x + \tan x| \\ \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \end{aligned}$$

and so the answer is $\boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}$

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