

Sample Problems

Compute each of the following integrals. Assume that a and b are positive.

1. $\int e^{-4x} dx$

2. $\int_0^8 e^{-4x} dx$

3. $\int (x^2 - 2)(x^3 - 6x)^{207} dx$

4. $\int e^{\cos x} \sin x dx$

5. $\int_0^{\pi/2} e^{\cos x} \sin x dx$

6. $\int \frac{3x}{(x^2 + 1)^7} dx$

7. $\int \frac{12x^3}{3x^4 + 1} dx$

8. $\int (-3x + 4)e^{-3x^2 + 8x} dx$

9. $\int_0^2 (-3x + 4)e^{-3x^2 + 8x} dx$

10. $\int \frac{x + 5}{x^2 + 1} dx$

11. $\int \frac{e^x}{e^x + 1} dx$

12. $\int \cos x \sin^5 x dx$

13. $\int \frac{x}{\sqrt{x+1}} dx$

14. $\int_0^{15} \frac{x}{\sqrt{x+1}} dx$

15. $\int_0^3 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

16. $\int \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx$

17. $\int_0^1 \frac{x + 1}{x^2 + 1} dx$

18. $\int_0^1 xe^{-3x^2} dx$

19. $\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} dx$

20. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

21. $\int \frac{x^3}{(x^2 + 2)^2} dx$

22. $\int \frac{1}{x^2 + 9} dx$

23. $\int \frac{1}{a^2x^2 + b^2} dx$

24. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$

25. $\int_0^{\pi/3} \sin 2x \sin x dx$

26. $\int 42 \cos x \sin x (\sin x + 1)^5 dx$

27. $\int_e^{e^2} \frac{1}{x \ln \sqrt{x}} dx$

28. $\int \tan x dx$

29. $\int \sec x dx$

30. $\int \frac{x^2}{\sqrt{1 - (x^3 - 1)^2}} dx$

31. $\int \frac{e^x}{e^{2x} + 1} dx$

32. $\int \frac{1}{\sqrt{16 - x^2}} dx$

33. $\int_4^5 \frac{1}{4x \ln x [\ln(\ln x)]^3} dx$

34. $\int_0^{\pi/3} \sin x (\cos x - \cos^3 x) dx$

35. $\int \sin^2 x dx$

36. $\int_{-1}^1 \sqrt{1 - x^2} dx$

37. $2 \int_{-r}^r \sqrt{r^2 - x^2} dx$

Sample Problems - Answers

- 1.) $-\frac{1}{4}e^{-4x} + C$ 2.) $\frac{1}{4} - \frac{1}{4}e^{-32}$ 3.) $\frac{1}{624}(x^3 - 6x)^{208} + C$ 4.) $-e^{\cos x} + C$ 5.) $e - 1$
- 6.) $-\frac{1}{4(x^2 + 1)^6} + C$ 7.) $\ln(3x^4 + 1) + C$ 8.) $\frac{1}{2}e^{-3x^2 + 8x} + C$ 9.) $\frac{1}{2}e^4 - \frac{1}{2}$
- 10.) $5 \tan^{-1} x + \frac{1}{2} \ln(x^2 + 1) + C$ 11.) $\ln(e^x + 1) + C$ 12.) $\frac{1}{6} \sin^6 x + C$ 13.) $\frac{2}{3}(x + 1)^{3/2} - 2(x + 1)^{1/2} + C$
- 14.) 36 15.) $\ln\left(e^3 + \frac{1}{e^3}\right) - \ln 2$ 16.) $2 \ln(\sqrt{x} + 1) + C$ 17.) $\frac{\pi}{4} + \frac{\ln 2}{2}$ 18.) $\frac{1}{6} - \frac{1}{6e^3}$
- 19.) $\frac{4}{3}(1 + \sqrt{x})\sqrt{1 + \sqrt{x}} + C$ 20.) $-2 \cos \sqrt{x} + C$ 21.) $\frac{1}{2} \ln(x^2 + 2) + \frac{1}{x^2 + 2} + C$ 22.) $\frac{1}{3} \tan^{-1} \frac{1}{3} x + C$
- 23.) $\frac{1}{ab} \tan^{-1}\left(\frac{a}{b}x\right) + C$ 24.) $\frac{1}{2}(\sin^{-1} x)^2 + C$ 25.) $\frac{\sqrt{3}}{4}$ 26.) $6(\sin x + 1)^7 - 7(\sin x + 1)^6 + C$
- 27.) $2 \ln 2$ 28.) $\ln|\sec x| + C$ 29.) $\ln|\sec x + \tan x| + C$ 30.) $\frac{1}{3} \sin^{-1}(x^3 - 1) + C$
- 31.) $\tan^{-1}(e^x) + C$ 32.) $\sin^{-1}\left(\frac{x}{4}\right) + C$ 33.) $\frac{1}{8 \ln^2(\ln 4)} - \frac{1}{8 \ln^2(\ln 5)}$ 34.) $\frac{9}{64}$
- 35.) $\frac{1}{2}x - \frac{1}{4} \sin 2x + C$ 36.) $\frac{\pi}{2}$ 37.) πr^2

Sample Problems - Solutions

Compute each of the following integrals. Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$.

$$1. \int e^{-4x} dx$$

Solution: Let $u = -4x$. Then $du = -4dx$ and so $dx = -\frac{1}{4}du$. We now substitute in the integral

$$\int e^{-4x} dx = \int e^u - \frac{1}{4}du = -\frac{1}{4} \int e^u du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{-4x} + C$$

$$2. \int_0^8 e^{-4x} dx$$

Solution:

$$\int_0^8 e^{-4x} dx = -\frac{1}{4}e^{-4x} \Big|_0^8 = -\frac{1}{4} \left(e^{-4(8)} - e^{-4(0)} \right) = -\frac{1}{4} (e^{-32} - e^0) = -\frac{1}{4} (e^{-32} - 1) = \frac{1}{4} - \frac{1}{4e^{32}}$$

$$3. \int (x^2 - 2)(x^3 - 6x)^{207} dx$$

Solution: Let $u = x^3 - 6x$. Then $du = (3x^2 - 6) dx$ and so $dx = \frac{1}{(3x^2 - 6)} du$. We now substitute in the integral

$$\begin{aligned} \int (x^2 - 2)(x^3 - 6x)^{207} dx &= \\ &= \int (x^2 - 2) u^{207} \frac{1}{(3x^2 - 6)} du = \int (x^2 - 2) u^{207} \frac{1}{3(x^2 - 2)} du = \frac{1}{3} \int u^{207} du = \frac{1}{3} \left(\frac{u^{208}}{208} \right) + C \\ &= \frac{1}{624} u^{208} + C = \frac{1}{624} (x^3 - 6x)^{208} + C \end{aligned}$$

$$4. \int e^{\cos x} \sin x dx$$

Solution: Let $u = \cos x$. Then $du = -\sin x dx$ and so $dx = -\frac{1}{\sin x} du$.

$$\int e^{\cos x} \sin x dx = \int e^u \sin x \frac{-1}{\sin x} du = - \int e^u du = -e^u + C = -e^{\cos x} + C$$

$$5. \int_0^{\pi/2} (e^{\cos x} \sin x) dx$$

Solution:

$$\int_0^{\pi/2} (e^{\cos x} \sin x) dx = -e^{\cos x} \Big|_0^{\pi/2} = - \left(e^{\cos(\pi/2)} - e^{\cos 0} \right) = - (e^0 - e^1) = e - 1$$

$$6. \int \frac{3x}{(x^2 + 1)^7} dx$$

Solution: Let $u = x^2 + 1$. Then $du = 2x dx$ and so $dx = \frac{1}{2x} du$.

$$\int \frac{3x}{(x^2 + 1)^7} dx = \int \frac{3x}{u^7} \frac{1}{2x} du = \frac{3}{2} \int \frac{1}{u^7} du = \frac{3}{2} \int u^{-7} du = \frac{3}{2} \frac{u^{-6}}{-6} + C = -\frac{1}{4u^6} + C = -\frac{1}{4(x^2 + 1)^6} + C$$

$$7. \int \frac{12x^3}{3x^4 + 1} dx$$

Solution: Let $u = 3x^4 + 1$. Then $du = 12x^3 dx$ and so $dx = \frac{1}{12x^3} du$.

$$\int \frac{12x^3}{3x^4 + 1} dx = \int \frac{12x^3}{u} \frac{1}{12x^3} du = \int \frac{1}{u} du = \ln |u| + C = \ln(3x^4 + 1) + C$$

$$8. \int (-3x + 4) e^{-3x^2 + 8x} dx$$

Solution: Let $u = -3x^2 + 8x$. Then $du = (-6x + 8) dx$ and so $dx = \frac{1}{(-6x + 8)} du$.

$$\begin{aligned} \int (-3x + 4) e^{-3x^2 + 8x} dx &= \int (-3x + 4) e^u \frac{1}{(-6x + 8)} du = \int (-3x + 4) e^u \frac{1}{2(-3x + 4)} du = \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C = \frac{1}{2} e^{-3x^2 + 8x} + C \end{aligned}$$

$$9. \int_0^2 (-3x + 4) e^{-3x^2 + 8x} dx$$

Solution:

$$\int_0^2 (-3x + 4) e^{-3x^2 + 8x} dx = \frac{1}{2} e^{-3x^2 + 8x} \Big|_0^2 = \frac{1}{2} \left(e^{-3(2)^2 + 8(2)} - e^{-3(0)^2 + 8(0)} \right) = \frac{1}{2} (e^4 - e^0) = \frac{1}{2} (e^4 - 1)$$

$$10. \int \frac{x + 5}{x^2 + 1} dx$$

Solution:

$$\int \frac{x + 5}{x^2 + 1} dx = \int \left(\frac{x}{x^2 + 1} + \frac{5}{x^2 + 1} \right) dx = \int \frac{x}{x^2 + 1} dx + \int \frac{5}{x^2 + 1} dx$$

These two integrals can be computed via very different methods. For the first integral, let $u = x^2 + 1$. Then $du = 2x dx$ and so $dx = \frac{1}{2x} du$.

$$\int \frac{x}{x^2 + 1} dx = \int \frac{x}{u} \frac{1}{2x} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + 1) + C$$

and the second integral is

$$\int \frac{5}{x^2 + 1} dx = 5 \int \frac{1}{x^2 + 1} dx = 5 \tan^{-1} x + C$$

and so

$$\begin{aligned} \int \frac{x + 5}{x^2 + 1} dx &= \int \frac{x}{x^2 + 1} dx + \int \frac{5}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C_1 + 5 \tan^{-1} x + C_2 \\ &= \frac{1}{2} \ln(x^2 + 1) + 5 \tan^{-1} x + C \end{aligned}$$

$$11. \int \frac{e^x}{e^x + 1} dx$$

Solution: Let $u = e^x + 1$. Then $du = e^x dx$ and so $dx = \frac{1}{e^x} du$.

$$\int \frac{e^x}{e^x + 1} dx = \int \frac{e^x}{u} \frac{1}{e^x} du = \int \frac{1}{u} du = \ln |u| + C = \ln(e^x + 1) + C$$

$$12. \int \cos x \sin^5 x dx$$

Solution: Let $u = \sin x$. Then $du = \cos x dx$ and so $dx = \frac{1}{\cos x} du$.

$$\int \cos x \sin^5 x dx = \int \cos x u^5 \frac{1}{\cos x} du = \int u^5 du = \frac{1}{6} u^6 + C = \frac{1}{6} \sin^6 x + C$$

$$13. \int \frac{x}{\sqrt{x+1}} dx$$

Solution: Let $u = x + 1$. Then $du = dx$ and $x = u - 1$

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}} dx &= \int \frac{u-1}{\sqrt{u}} du = \int (u-1) u^{-1/2} du = \int uu^{-1/2} - u^{-1/2} du = \int u^{1/2} - u^{-1/2} du \\ &= \frac{2}{3} u^{3/2} - 2u^{1/2} + C = \frac{2}{3} (x+1)^{3/2} - 2(x+1)^{1/2} + C \end{aligned}$$

$$14. \int_0^{15} \frac{x}{\sqrt{x+1}} dx$$

Solution: We worked out the indefinite integral in the previous problem.

$$\begin{aligned} \int_0^{15} \frac{x}{\sqrt{x+1}} dx &= \left(\frac{2}{3} (x+1)^{3/2} - 2(x+1)^{1/2} \right) \Big|_0^{15} \\ &= \left(\frac{2}{3} (15+1)^{3/2} - 2(15+1)^{1/2} \right) - \left(\frac{2}{3} (0+1)^{3/2} - 2(0+1)^{1/2} \right) \\ &= \left(\frac{2}{3} (16^{3/2}) - 2(16^{1/2}) \right) - \left(\frac{2}{3} (1^{3/2}) - 2(1^{1/2}) \right) = \left(\frac{2}{3} (64) - 2(4) \right) - \left(\frac{2}{3} - 2 \right) \\ &= \frac{128}{3} - 8 - \frac{2}{3} + 2 = \frac{126}{3} - 6 = 36 \end{aligned}$$

$$15. \int_0^3 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

Solution: Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x}) dx$ and so $dx = \frac{1}{(e^x - e^{-x})} du$. Also, when $x = 0$, then $u = e^0 + e^{-0} = 2$ and when $x = 3$, then $u = e^3 + e^{-3}$

$$\int_0^3 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int_2^{e^3 + e^{-3}} \frac{e^x - e^{-x}}{u} \frac{1}{(e^x - e^{-x})} du = \int_2^{e^3 + e^{-3}} \frac{1}{u} du = \ln |u| \Big|_2^{e^3 + e^{-3}} = \ln(e^3 + e^{-3}) - \ln 2$$

$$16. \int \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$$

Solution: Let $u = \sqrt{x} + 1$. Then $du = \frac{1}{2\sqrt{x}}dx$ and so $dx = 2\sqrt{x}du$.

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx = \int \frac{1}{\sqrt{x}u} 2\sqrt{x}du = 2 \int \frac{1}{u} du = 2 \ln |u| + C = 2 \ln(\sqrt{x}+1) + C$$

$$17. \int_0^1 \frac{x+1}{x^2+1} dx$$

Solution: We will first work out the indefinite integral. (For more details, please refer to problem #10.)

$$\int \frac{x+1}{x^2+1} dx = \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C$$

and so the definite integral is.

$$\begin{aligned} \int_0^1 \frac{x+1}{x^2+1} dx &= \left(\frac{1}{2} \ln(x^2+1) + \tan^{-1} x \right) \Big|_0^1 = \left(\frac{1}{2} \ln(1^2+1) + \tan^{-1} 1 \right) - \left(\frac{1}{2} \ln(0^2+1) + \tan^{-1} 0 \right) \\ &= \left(\frac{1}{2} \ln 2 + \frac{\pi}{4} \right) - 0 = \frac{1}{2} \ln 2 + \frac{\pi}{4} \end{aligned}$$

$$18. \int_0^1 x e^{-3x^2} dx$$

Solution: We will first work out the indefinite integral. Let $u = -3x^2$. Then $du = -6x dx$ and so $dx = -\frac{1}{6x} du$. We will also have to substitute the limits of the integration: when $x = 0$, then $u = 0$ and when $x = 1$, then $u = -3$.

$$\int_0^1 x e^{-3x^2} dx = \int_0^{-3} x e^u \left(-\frac{1}{6x} du \right) = -\frac{1}{6} \int_0^{-3} e^u du = \frac{1}{6} \int_{-3}^0 e^u du = \frac{1}{6} e^u \Big|_{-3}^0 = \frac{1}{6} (e^0 - e^{-3}) = \frac{1}{6} \left(1 - \frac{1}{e^3} \right)$$

$$19. \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = \frac{4}{3} (1+\sqrt{x}) \sqrt{1+\sqrt{x}} + C$$

Solution: We will use substitution. Let $u = 1 + \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}} dx$ and so $dx = 2\sqrt{x} du$

$$\begin{aligned} \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx &= \int \frac{\sqrt{u}}{\sqrt{x}} 2\sqrt{x} du = 2 \int \sqrt{u} du = 2 \left(\frac{2}{3} \right) u^{3/2} + C = \frac{4}{3} (1+\sqrt{x})^{3/2} + C \\ &= \frac{4}{3} (1+\sqrt{x}) \sqrt{1+\sqrt{x}} + C = \frac{4}{3} \sqrt{1+\sqrt{x}} + \frac{4}{3} \sqrt{x} \sqrt{1+\sqrt{x}} + C \end{aligned}$$

$$20. \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Solution: Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$ and so $dx = 2\sqrt{x} du$.

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \frac{\sin u}{\sqrt{x}} 2\sqrt{x} du = 2 \int \sin u du = -2 \cos u + C = -2 \cos \sqrt{x} + C$$

$$21. \int \frac{x^3}{(x^2 + 2)^2} dx$$

Solution: This integral is an interesting case of substitution. Let $u = x^2 + 2$. Then $du = 2x dx$ and so $dx = \frac{du}{2x}$ and also, $x^2 = u - 2$.

$$\int \frac{x^3}{(x^2 + 2)^2} dx = \int \frac{x^3}{u^2} \frac{du}{2x} = \frac{1}{2} \int \frac{x^2}{u^2} du$$

It looks like we are 'stuck' with both x and u in the integrand. However, we can get rid of x^2 because it is $u - 2$.

$$\frac{1}{2} \int \frac{x^2}{u^2} du = \frac{1}{2} \int \frac{u - 2}{u^2} du = \frac{1}{2} \int \frac{1}{u} - \frac{2}{u^2} du = \frac{1}{2} \ln |u| + \frac{1}{u} + C = \frac{1}{2} \ln(x^2 + 2) + \frac{1}{x^2 + 2} + C$$

Note that because $x^2 + 2$ is positive for all values of x , the expression $\ln|x^2 + 2|$ could be simplified to $\ln(x^2 + 2)$.

$$22. \int \frac{1}{x^2 + 9} dx$$

Solution: The idea here is that we will 'get rid' of the 9 by factoring it out. If we used a substitution that would turn x^2 into $9u^2$, then the integrand would be $\frac{1}{9u^2 + 9} = \frac{1}{9} \cdot \frac{1}{u^2 + 1}$. We will pursue this substitution: $x^2 = 9u^2 \implies x = \pm 3u$. We select one of these.

Let $u = \frac{1}{3}x$. Then $du = \frac{1}{3}dx$ and so $dx = 3du$.

$$\int \frac{1}{x^2 + 9} dx = \int \frac{1}{9u^2 + 9} 3du = \frac{3}{9} \int \frac{1}{u^2 + 9} du = \frac{1}{3} \arctan u + C = \frac{1}{3} \arctan \frac{x}{3} + C$$

$$23. \int \frac{1}{a^2x^2 + b^2} dx$$

Solution: We need a substitution under which $a^2x^2 = b^2u^2$. This would be convenient because then

$$\frac{1}{a^2x^2 + b^2} = \frac{1}{b^2u^2 + b^2} = \frac{1}{b^2} \cdot \frac{1}{u^2 + 1}$$

So we will pursue this substitution. We solve $a^2x^2 = b^2u^2$ for a possible value of u and obtain $u = \frac{a}{b}x$.

Then $du = \frac{a}{b}dx$ and so $\frac{b}{a}du = dx$.

$$\begin{aligned} \int \frac{1}{a^2x^2 + b^2} dx &= \int \frac{1}{b^2u^2 + b^2} \left(\frac{b}{a}du\right) = \int \frac{1}{b^2} \cdot \frac{1}{u^2 + 1} \cdot \frac{b}{a} du = \frac{b}{ab^2} \int \frac{1}{u^2 + 1} du = \frac{1}{ab} \tan^{-1} u + C \\ &= \frac{1}{ab} \tan^{-1} \left(\frac{a}{b}x\right) + C \end{aligned}$$

$$24. \int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$$

Solution: Let $u = \sin^{-1} x$. We differentiate both sides and obtain that $du = \frac{1}{\sqrt{1 - x^2}} dx$ and so $dx = \sqrt{1 - x^2} du$

$$\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx = \int \frac{u}{\sqrt{1 - x^2}} \sqrt{1 - x^2} du = \int u du = \frac{u^2}{2} + C = \frac{1}{2} (\sin^{-1} x)^2 + C$$

$$25. \int_0^{\pi/3} \sin 2x \sin x \, dx$$

Solution:

$$\int_0^{\pi/3} \sin 2x \sin x \, dx = \int_0^{\pi/3} (2 \sin x \cos x) \sin x \, dx = 2 \int_0^{\pi/3} \sin^2 x \cos x \, dx$$

From here we will use substitution. Let $u = \sin x$ then $du = \cos x \, dx$. We will also need to compute the limits of the integral using u instead of x . When $x = 0$, then $u = \sin 0 = 0$ and when $x = \frac{\pi}{3}$, then

$$u = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

$$2 \int_0^{\pi/3} \sin^2 x \cos x \, dx = 2 \int_0^{\sqrt{3}/2} u^2 \, du = \frac{2}{3} u^3 \Big|_0^{\sqrt{3}/2} = \frac{2}{3} \left(\frac{\sqrt{3}}{2} \right)^3 = \frac{2}{3} \cdot \frac{3\sqrt{3}}{4 \cdot 2} = \frac{\sqrt{3}}{4}$$

$$26. \int 42 \cos x \sin x (\sin x + 1)^5 \, dx$$

Solution: We will use substitution. Let $u = \sin x + 1$. then $du = \cos x \, dx$ and so $\frac{du}{\cos x} = dx$. Also, $u - 1 = \sin x$.

$$\begin{aligned} \int 42 \cos x \sin x (\sin x + 1)^5 \, dx &= \int 42 \cos x (u - 1) u^5 \frac{du}{\cos x} = 42 \int (u - 1) u^5 \, du = 42 \int u^6 - u^5 \, du \\ &= 42 \left(\frac{u^7}{7} - \frac{u^6}{6} \right) + C = 6u^7 - 7u^6 + C = 6(\sin x + 1)^7 - 7(\sin x + 1)^6 + C \end{aligned}$$

$$27. \int_e^{e^2} \frac{1}{x \ln \sqrt{x}} \, dx$$

Solution:

$$\int_e^{e^2} \frac{1}{x \ln \sqrt{x}} \, dx = \int_e^{e^2} \frac{1}{x \left(\frac{1}{2} \right) \ln x} \, dx = \int_e^{e^2} \frac{2}{x \ln x} \, dx = 2 \int_e^{e^2} \frac{1}{x \ln x} \, dx$$

Let $u = \ln x$. Then $du = \frac{1}{x} dx$. We will also have to substitute the limits of the integration: when $x = e$, then $u = \ln e = 1$ and when $x = e^2$, then $u = \ln e^2 = 2$.

$$2 \int_e^{e^2} \frac{1}{x \ln x} \, dx = 2 \int_e^{e^2} \frac{1}{\ln x} \frac{1}{x} \, dx = 2 \int_1^2 \frac{1}{u} \, du = 2 \int_1^2 \frac{1}{u} \, du = 2 \ln |u| \Big|_1^2 = 2(\ln 2 - \ln 1) = 2 \ln 2$$

$$28. \int \tan x \, dx$$

Solution: Let $u = \cos x$, then $du = -\sin x \, dx$ and so $dx = -\frac{du}{\sin x}$

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{\sin x}{u} \left(-\frac{du}{\sin x} \right) = -\int \frac{1}{u} \, du = -\ln |u| + C = -\ln |\cos x| + C \\ &= \ln |(\cos x)^{-1}| + C = \ln |\sec x| + C \end{aligned}$$

$$29. \int \sec x \, dx$$

$$\text{Solution: } \int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

From here we will use substitution. Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x) \, dx$ and so $dx = \frac{du}{\sec^2 x + \sec x \tan x}$

$$\begin{aligned} \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{u} \left(\frac{du}{\sec^2 x + \sec x \tan x} \right) = \int \frac{1}{u} \, du = \ln |u| + C \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

$$30. \int \frac{x^2}{\sqrt{1 - (x^3 - 1)^2}} \, dx$$

Solution: Let $u = x^3 - 1$. Then $du = 3x^2 \, dx$ and so $dx = \frac{du}{3x^2}$

$$\int \frac{x^2}{\sqrt{1 - (x^3 - 1)^2}} \, dx = \int \frac{x^2}{\sqrt{1 - u^2}} \left(\frac{du}{3x^2} \right) = \frac{1}{3} \int \frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{3} \arcsin u + C = \frac{1}{3} \arcsin (x^3 - 1) + C$$

$$31. \int \frac{e^x}{e^{2x} + 1} \, dx$$

Solution: Let $u = e^x$. Then $du = e^x \, dx$.

$$\int \frac{e^x}{e^{2x} + 1} \, dx = \int \frac{1}{u^2 + 1} e^x \, dx = \int \frac{1}{u^2 + 1} \, du = \arctan u + C = \arctan (e^x) + C$$

$$32. \int \frac{1}{\sqrt{16 - x^2}} \, dx$$

Solution: The basic integral we see here is $\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C$. We will use substitution to get rid of the 16 in the denominator. Let us substitute u so that $x^2 = 16u^2$. Then $x = 4u$ and $dx = 4du$

$$\int \frac{1}{\sqrt{16 - x^2}} \, dx = \int \frac{1}{\sqrt{16 - 16u^2}} (4du) = \int \frac{4}{\sqrt{16}\sqrt{1 - u^2}} \, du = \int \frac{1}{\sqrt{1 - u^2}} \, du = \sin^{-1} u + C = \sin^{-1} \left(\frac{x}{4} \right) + C$$

$$33. \int_4^5 \frac{1}{4x \ln x [\ln(\ln x)]^3} \, dx$$

Solution: Note that $\ln^3 \ln x$ is short for $[\ln(\ln x)]^3$. We will substitute $u = \ln(\ln x)$. Then $du = \frac{1}{\ln x} \cdot \frac{1}{x} \, dx$. We also substitute the limits of the integral: if $x = 4$, then $u = \ln(\ln 4)$ and if $x = 5$, then $u = \ln(\ln 5)$. Then

$$\begin{aligned} I &= \int_4^5 \frac{1}{4x \ln x [\ln(\ln x)]^3} \, dx = \int_4^5 \frac{1}{4 [\ln(\ln x)]^3} \left(\frac{1}{x \ln x} \, dx \right) = \int_{\ln \ln 4}^{\ln \ln 5} \frac{1}{4u^3} \, du = \frac{1}{4} \int_{\ln \ln 4}^{\ln \ln 5} u^{-3} \, du \\ &= \frac{1}{4} \frac{u^{-2}}{-2} \Big|_{\ln \ln 4}^{\ln \ln 5} = -\frac{1}{8} \left(\frac{1}{(\ln \ln 5)^2} - \frac{1}{(\ln \ln 4)^2} \right) = \frac{1}{8 \ln^2(\ln 4)} - \frac{1}{8 \ln^2(\ln 5)} \end{aligned}$$

$$34. \int_0^{\pi/3} \sin x (\cos x - \cos^3 x) dx$$

Solution: We first simplify the integrand.

$$\int \sin x (\cos x - \cos^3 x) dx = \int \sin x \cos x (1 - \cos^2 x) dx = \int \sin x \cos x \sin^2 x dx = \int \sin^3 x \cos x dx$$

Let $u = \sin x$. Then $du = \cos x dx$. We will also need to compute the limits of the integral using u instead of x . When $x = 0$, then $u = \sin 0 = 0$ and when $x = \frac{\pi}{3}$, then $u = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$\int_0^{\pi/3} \sin^3 x \cos x dx = \int_0^{\sqrt{3}/2} u^3 \cos x dx = \int_0^{\sqrt{3}/2} u^3 du = \frac{u^4}{4} \Big|_0^{\sqrt{3}/2} = \frac{1}{4} \left(\frac{\sqrt{3}}{2} \right)^4 = \frac{1}{4} \cdot \frac{9}{16} = \frac{9}{64}$$

$$35. \int \sin^2 x dx$$

Solution: Recall the double-angle formula for cosine and solve it for $\sin^2 x$.

$$\cos 2x = 1 - 2 \sin^2 x \quad \implies \quad \sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\int \sin^2 x dx = \int \frac{1}{2} (1 - \cos 2x) dx = \frac{1}{2} \int 1 - \cos 2x dx = \frac{1}{2} \left(\int 1 dx - \int \cos 2x dx \right)$$

For the second integral, we substitute $u = 2x$ and $du = 2dx$.

$$\int \cos 2x dx = \int \cos u \frac{du}{2} = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin 2x + C$$

Thus the entire integral is

$$\begin{aligned} \int \sin^2 x dx &= \frac{1}{2} \left(\int 1 dx - \int \cos 2x dx \right) = \frac{1}{2} \left(x + C_1 - \left(\frac{1}{2} \sin 2x + C_2 \right) \right) = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x + C_3 \right) \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \end{aligned}$$

$$36. \int_{-1}^1 \sqrt{1-x^2} dx$$

Solution: Let us first notice that the function is even and the limits of the integral are opposites.

$$\int_{-1}^1 \sqrt{1-x^2} dx = 2 \int_0^1 \sqrt{1-x^2} dx$$

This will make the computation somewhat easier. The following method is called trigonometric substitution. Let θ be so that $x = \cos \theta$. Then $\theta = \cos^{-1} x$. The substitution $\theta = \cos^{-1} x$ means that $x = \cos \theta$ and that θ is in the interval $[0, \pi]$. This is important to know because on $[0, \pi]$ $\sin \theta$ is non-negative and so $|\sin \theta| = \sin \theta$. We differentiate the statement $x = \cos \theta$ and obtain $dx = -\sin \theta d\theta$. We will also need to substitute the limits of integration. When $x = 0$, then $\theta = \cos^{-1} 0 = \frac{\pi}{2}$ and when $x = 1$, then $\theta = \cos^{-1} 1 = 0$.

$$\begin{aligned}
2 \int_0^1 \sqrt{1-x^2} dx &= 2 \int_{\pi/2}^0 \sqrt{1-\cos^2 \theta} (-\sin \theta) d\theta = -2 \int_{\pi/2}^0 \sqrt{\sin^2 \theta} \sin \theta d\theta = -2 \int_{\pi/2}^0 |\sin \theta| \sin \theta d\theta \\
&= -2 \int_{\pi/2}^0 \sin^2 \theta d\theta = 2 \int_0^{\pi/2} \sin^2 \theta d\theta
\end{aligned}$$

In the previous problem we have computed the indefinite integral:

$$\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

Thus the definite integral is

$$\begin{aligned}
2 \int_0^{\pi/2} \sin^2 u du &= 2 \left(\frac{1}{2}u - \frac{1}{4}\sin 2u \right) \Big|_0^{\pi/2} = \left(u - \frac{1}{2}\sin 2u \right) \Big|_0^{\pi/2} = \left(\frac{\pi}{2} - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right) \right) - \left(0 - \frac{1}{2}\sin 2(0) \right) \\
&= \left(\frac{\pi}{2} - \frac{1}{2}\sin \pi \right) - \left(0 - \frac{1}{2}\sin 0 \right) = \left(\frac{\pi}{2} - \frac{1}{2}(0) \right) - 0 = \frac{\pi}{2}
\end{aligned}$$

Thus

$$\int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2} \quad \implies \quad 2 \int_{-1}^1 \sqrt{1-x^2} dx = \pi$$

This is one of the definitions of π in modern mathematics. Also notice that we have just proved the area formula for the unit circle.

$$37. \quad 2 \int_{-r}^r \sqrt{r^2-x^2} dx$$

Solution: First we notice that this function is even and the limits of integration are opposites:

$$2 \int_{-r}^r \sqrt{r^2-x^2} dx = 4 \int_0^r \sqrt{r^2-x^2} dx$$

We will apply a trigonometric substitution under which $x^2 = r^2 \cos^2 \theta$. This will be useful because $\sqrt{r^2-x^2}$ will become

$$\sqrt{r^2-x^2} = \sqrt{r^2-r^2 \cos^2 \theta} = \sqrt{r^2(1-\cos^2 \theta)} = \sqrt{r^2} \sqrt{\sin^2 \theta} = r |\sin \theta|$$

So, we solve for a possible value of θ in $x^2 = r^2 \cos^2 \theta$.

$$\begin{aligned}
x^2 &= r^2 \cos^2 \theta \\
x &= r \cos \theta \\
\frac{x}{r} &= \cos \theta \\
\cos^{-1} \left(\frac{x}{r} \right) &= \theta
\end{aligned}$$

So our substitution is $\theta = \cos^{-1} \left(\frac{x}{r} \right)$. That is the same as $\cos \theta = \frac{x}{r}$ and θ is in the interval $[0, \pi]$ (the range of the function \cos^{-1}). This is useful to know because on the interval $[0, \pi]$, sine is always non-negative and

so $|\sin \theta|$ can be simplified as $\sin \theta$. We also differentiate both sides of the equation $\cos \theta = \frac{x}{r}$ and obtain that $-\sin \theta d\theta = \frac{dx}{r}$ and so $dx = -r \sin \theta d\theta$. We will need to substitute the limits as well. When $x = 0$, then $\theta = \cos^{-1} \left(\frac{0}{r} \right) = \cos^{-1} 0 = \frac{\pi}{2}$ and when $x = r$, then $\theta = \cos^{-1} \left(\frac{r}{r} \right) = \cos^{-1} 1 = 0$. We are now ready to integrate:

$$\begin{aligned} I &= 4 \int_0^r \sqrt{r^2 - x^2} dx = 4 \int_{\pi/2}^0 \sqrt{r^2 - r^2 \cos^2 \theta} (-r \sin \theta d\theta) = 4 \int_{\pi/2}^0 -r \sin \theta \sqrt{r^2 (1 - \cos^2 \theta)} d\theta = \\ &= 4 \int_{\pi/2}^0 -r \sin \theta r \sqrt{\sin^2 \theta} d\theta = 4 \int_{\pi/2}^0 -r^2 \sin \theta |\sin \theta| d\theta = -4r^2 \int_{\pi/2}^0 \sin \theta \cdot \sin \theta d\theta \\ &= -4r^2 \int_{\pi/2}^0 \sin^2 \theta d\theta = 4r^2 \int_0^{\pi/2} \sin^2 \theta d\theta \end{aligned}$$

We have already computed the antiderivative of $\sin^2 \theta$ in problem 35:

$$\int \sin^2 \theta d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\begin{aligned} I &= 4r^2 \int_0^{\pi/2} \sin^2 \theta d\theta = 4r^2 \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \Big|_0^{\pi/2} = r^2 (2\theta - \sin 2\theta) \Big|_0^{\pi/2} \\ &= r^2 \left[\left(2 \left(\frac{\pi}{2} \right) - \sin 2 \left(\frac{\pi}{2} \right) \right) - (2(0) - \sin 2(0)) \right] = r^2 [(\pi - \sin \pi) - (0 - \sin 0)] = r^2 [(\pi - 0) - 0] = \pi r^2 \end{aligned}$$